

# Hankel determinant and orthogonal polynomials for a Gaussian weight with a discontinuity at the edge

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## Abstract

We compute asymptotics for Hankel determinants and orthogonal polynomials with respect to a discontinuous Gaussian weight, in a critical regime where the discontinuity is close to the edge of the associated equilibrium measure support. Their behavior is described in terms of the Ablowitz-Segur family of solutions to the Painlevé II equation. Our results complement the ones in [33]. As consequences of our results, we conjecture asymptotics for an Airy kernel Fredholm determinant and total integral identities for Painlevé II transcendents, and we also prove a new result on the poles of the Ablowitz-Segur solutions to the Painlevé II equation. We also highlight applications of our results in random matrix theory.

## 1 Introduction

Consider the Hankel determinant,

$$H_n(\lambda_0, \beta) = \det \left( \int_{-\infty}^{\infty} x^{j+k} w(x) dx \right)_{j,k=0}^{n-1} = \frac{1}{n!} \int \cdots \int_{-\infty}^{\infty} \prod_{i < j} (x_i - x_j)^2 \prod_{k=1}^n w(x_k) dx_k, \quad (1.1)$$

with respect to a discontinuous Gaussian weight of the form

$$w(x) = e^{-x^2} \times \begin{cases} e^{\pi i \beta}, & x < \lambda_0 \\ e^{-\pi i \beta}, & x \geq \lambda_0 \end{cases}, \quad \operatorname{Re} \beta \in \left( -\frac{1}{2}, \frac{1}{2} \right], \quad \lambda_0 \in \mathbb{R}. \quad (1.2)$$

The weight is periodic in  $\beta$  and we can restrict to the case  $-1/2 < \operatorname{Re} \beta \leq 1/2$  without loss of generality. If  $\beta$  is purely imaginary, the weight is positive.

We also consider the monic orthogonal polynomials  $p_n$  of degree  $n$  with respect to the weight  $w(x)$  on the real line, defined by the orthogonality conditions

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) w(x) dx = h_n \delta_{nm}, \quad h_n = h_n(\lambda_0, \beta). \quad (1.3)$$

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Those are connected to the Hankel determinant  $H_n$  by the well-known identity  $H_n(\lambda_0, \beta) = \prod_{k=0}^{n-1} h_k(\lambda_0, \beta)$ . We denote by  $R_n = R_n(\lambda_0, \beta)$  and  $Q_n = Q_n(\lambda_0, \beta)$  the recurrence coefficients in the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + Q_n p_n(x) + R_n p_{n-1}(x). \quad (1.4)$$

The question which we are concerned with in this paper is the large  $n$  behavior of the Hankel determinants  $H_n$ , the polynomials  $p_n(x)$ , and their recurrence coefficients  $R_n$  and  $Q_n$ , in the regime where the point of discontinuity  $\lambda_0$  behaves like  $\sqrt{2n}$ . They can asymptotically be expressed in terms of the Ablowitz-Segur solutions to the Painlevé II equation. As important by-products of the asymptotics for the Hankel determinants, we also conjecture so-called large gap asymptotics for an Airy kernel Fredholm determinant and total integral identities for the Ablowitz-Segur solutions of the Painlevé II equation. Relying on a result of [33], we in addition prove a new result about the poles for those Painlevé transcendents.

If we let  $\lambda_0 = \lambda \sqrt{2n}$ , the large  $n$  asymptotics of the orthogonal polynomials (1.3), the recurrence coefficients (1.4), and the Hankel determinant (1.1) depend dramatically on whether  $|\lambda| < 1$  or  $|\lambda| > 1$ , i.e. whether the jump location  $\lambda_0$  is inside or outside of the support  $[-\sqrt{2n}, \sqrt{2n}]$  of the equilibrium measure with Gaussian external field. In the case  $|\lambda| > 1$ , all the objects of interest behave effectively as they do for the pure Gaussian weight (i.e., the case where we formally set  $\lambda_0 = +\infty$ ); the discontinuity yields an exponentially small correction only [25]. In the case  $|\lambda| < 1$ , the situation is different; the discontinuity of the weight becomes strongly visible in the large  $n$  behavior of the orthogonal polynomials, the recurrence coefficients, and the Hankel determinant [21]. For the Hankel determinant, it was proved in [21, equation (1.5)] that

$$H_n(\lambda_0, \beta) = H_n(\lambda_0, 0) G(1 + \beta)G(1 - \beta)(1 - \lambda^2)^{-3\beta^2/2}(8n)^{-\beta^2} \times \\ \times \exp\left(2in\beta\left(\arcsin \lambda + \lambda \sqrt{1 - \lambda^2}\right)\right) \left(1 + O\left(\frac{\log n}{n^{1-4|\operatorname{Re} \beta|}}\right)\right), \quad |\operatorname{Re} \beta| < \frac{1}{4}, \quad (1.5)$$

as  $n \rightarrow \infty$ , uniformly for  $\lambda$  in compact subsets of  $(-1, 1)$ . Here  $G$  is the Barnes'  $G$ -function, and

$$H_n(\lambda_0, 0) = (2\pi)^{n/2} 2^{-n^2/2} \prod_{k=1}^{n-1} k! \quad (1.6)$$

denotes the Hankel determinant corresponding to the pure Gaussian weight  $e^{-x^2}$ . Asymptotics for the recurrence coefficients  $Q_n$  and  $R_n$  in the case  $-1 < \lambda < 1$  are also given in [21].

In this paper, we analyze the transition regime where the point  $\lambda_0$  of discontinuity of the weight is (relatively) close to  $\sqrt{2n}$ . More precisely we let

$$\lambda_0 = \lambda \sqrt{2n}, \quad \lambda = 1 + \frac{t}{2} n^{-2/3}, \quad (1.7)$$

where  $t \in \mathbb{R}$ . We will see that the asymptotic behavior of  $H_n$ ,  $p_n$ ,  $R_n$ , and  $Q_n$  depends in a non-trivial way on the parameter  $t$  in (1.7). The asymptotic behavior is described in terms of a family of solutions to the Painlevé II equation

$$u_{tt} = tu + 2u^3, \quad (1.8)$$

with the asymptotic behavior

$$u(t; \kappa) \sim \kappa \operatorname{Ai}(t), \quad t \rightarrow +\infty, \quad (1.9)$$

where  $\operatorname{Ai}$  denotes the Airy function, and

$$u(t; \kappa) = \frac{1}{(-t)^{1/4}} \sqrt{2i\beta} \sin \phi(t; \beta) + O\left(\frac{1}{t^{2-3|\operatorname{Re} \beta|}}\right), \quad t \rightarrow -\infty, \quad (1.10)$$

with

$$\phi(t; \beta) = -\frac{\pi}{4} - i \log \frac{\Gamma(-\beta)}{\Gamma(\beta)} + \frac{2}{3}(-t)^{3/2} - \frac{3}{2}i\beta \log(-t) - 3i\beta \log 2, \quad \kappa^2 = 1 - e^{-2\pi i\beta}, \quad |\operatorname{Re} \beta| < \frac{1}{2}. \quad (1.11)$$

For  $0 < \kappa < 1$ , these solutions are known as the Ablowitz-Segur solutions [1] of the second Painlevé equation. They are uniquely characterized either by (1.9) or by (1.10). Moreover, it is known that  $u(t; \kappa)$  has no singularities for  $t$  on the real line if  $\kappa \in i\mathbb{R}$  or if  $|\kappa| < 1$ . For  $\kappa \in \mathbb{R} \setminus [-1, 1]$ , or equivalently  $|\operatorname{Re} \beta| = 1/2$ , it is known that  $u(\tau; \kappa)$  does have real poles [4]. Relying on a result from [33], we will prove the following result, stating that  $u$  has no real poles for any  $\kappa \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$ , or equivalently for any  $\beta$  with  $|\operatorname{Re} \beta| < 1/2$ .

**Theorem 1.** *Let  $u(t; \kappa)$  be the solution to the Painlevé II equation (1.8) characterized by (1.9). If  $\kappa \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$  is fixed, then  $u(t; \kappa)$  has no poles at real values of  $t$ .*

In the case  $\kappa = 0$ , we simply have  $u(t; \kappa) = 0$ ; the unique Painlevé II solution satisfying (1.9) with  $\kappa = \pm 1$  (which means formally that  $\beta = -i\infty$ ) is known as the Hastings-McLeod solution.

The function  $y(t; \beta) = u(t; \kappa)^2$  solves the Painlevé XXXIV equation

$$y_{tt} = 4y^2 + 2ty + \frac{(y_t)^2}{2y}. \quad (1.12)$$

The function  $y(t, \beta)$  and equation (1.12) are, in fact, the objects which directly appear in our double scaling analysis of  $H_n$ ,  $p_n$ ,  $R_n$  and  $Q_n$ . Our next result describes the asymptotics of the Hankel determinants  $H_n(\lambda_0, \beta)$ .

**Theorem 2.** *Let  $|\operatorname{Re} \beta| < 1/2$  and let  $H_n(\lambda_0, \beta)$  be the Hankel determinant (1.1) corresponding to the weight (1.2), with  $\lambda_0$  given by (1.7). If  $\kappa^2 = 1 - e^{-2\pi i\beta}$ , we have*

$$H_n(\lambda_0, \beta) = e^{i\pi\beta n} H_n(\lambda_0, 0) \exp\left(-\int_t^\infty (\tau - t)u(\tau; \kappa)^2 d\tau\right) (1 + o(1)), \quad n \rightarrow \infty, \quad (1.13)$$

uniformly for  $t \in [-M, \infty)$  for any  $M > 0$  and for  $\beta$  in compact subsets of  $|\operatorname{Re} \beta| < 1/2$ , where  $H_n(\lambda_0, 0)$  is given in (1.6).

Theorem 2 has two consequences which are not directly related to the Hankel determinants or orthogonal polynomials studied in this paper, but which are of independent interest. To describe them, we note first that the exponential in (1.13) can be recognized as the Tracy-Widom formula for the Fredholm determinant  $\det(1 - \kappa^2 K_{\text{Ai}}|_{[t, +\infty)})$ , where  $K_{\text{Ai}}|_{[t, +\infty)}$  is the integral operator with kernel

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y} \quad (1.14)$$

acting on  $[t, +\infty)$ . Indeed, it was shown in [31] that

$$\det(1 - \kappa^2 K_{\text{Ai}}|_{[t, +\infty)}) = \exp\left(-\int_t^\infty (\tau - t)u(\tau; \kappa)^2 d\tau\right). \quad (1.15)$$

This observation, together with a strengthened version of the Hankel determinant asymptotics (1.5), allows us to formulate the following conjecture about the  $t \rightarrow -\infty$  asymptotics of  $\det(1 - \kappa^2 K_{\text{Ai}}|_{[t, +\infty)})$ .

**Conjecture 3.** Let  $\kappa \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$  and define  $\beta$  by

$$\kappa^2 = 1 - e^{-2\pi i\beta}, \quad |\operatorname{Re} \beta| < 1/4. \quad (1.16)$$

As  $t \rightarrow -\infty$ , we have

$$\log \det \left( 1 - \kappa^2 K_{\text{Ai}} \Big|_{[t, +\infty)} \right) = -\frac{4}{3} i\beta (-t)^{3/2} - \frac{3}{2} \beta^2 \log(-t) + \log(G(1+\beta)G(1-\beta)) - 3\beta^2 \log 2 + o(1), \quad (1.17)$$

or equivalently in form of a total integral identity

$$\lim_{t \rightarrow -\infty} \left( - \int_t^\infty (\tau - t) u(\tau; \kappa)^2 d\tau + \frac{4}{3} i\beta (-t)^{3/2} + \frac{3}{2} \beta^2 \log(-t) \right) = \log(G(1+\beta)G(1-\beta)) - 3\beta^2 \log 2. \quad (1.18)$$

*Remark.* Similar asymptotics for the Airy kernel determinant in the case  $\kappa = 1$  were proved in [2, 12]: we then have

$$\log \det \left( I - K_{\text{Ai}} \Big|_{[t, +\infty)} \right) = \frac{t^3}{12} - \frac{1}{8} \log |t| + c_0 + O(t^{-3}), \quad t \rightarrow -\infty, \quad (1.19)$$

where  $c_0 = \log 2/24 + \zeta'(-1)$  and  $\zeta$  is the Riemann  $\zeta$ -function. As  $\kappa \rightarrow 1$ , it was shown recently in [7] that

$$\log \det \left( I - \kappa^2 K_{\text{Ai}} \Big|_{[t, +\infty)} \right) = \frac{t^3}{12} - \frac{1}{8} \log |t| + c_0 + o(1), \quad t \rightarrow -\infty, \quad (1.20)$$

as long as  $\kappa < 1$ , and  $\kappa \rightarrow 1$  sufficiently rapidly so that

$$- \frac{\log(1 - \kappa^2)}{(-t)^{3/2}} > \frac{2\sqrt{2}}{3}. \quad (1.21)$$

The total integrals of different expressions involving the second Painlevé transcendent were studied in [3]. The integral (1.18) does not belong to the type which can be handled by the technique of [3]. Indeed, like the similar integral corresponding to equation (1.19), the integral in (1.18) belongs to the third, most difficult type of total integrals of Painlevé functions as classified in the end of Section 6 of [3]. This means that the evaluation of this integral goes beyond the analysis of the Riemann-Hilbert problem corresponding to the Ablowitz-Segur Painlevé II transcendent. As we already indicated, the proof of (1.18) can be achieved via an improvement of the error term in (1.5). Another possibility is to use certain differential identities for the Airy determinant in (1.15) with respect to  $\kappa$ . We intend to consider these issues in our next publication.

Additionally, the asymptotics of the PXXXIV transcendent  $y(t; \beta) = u(t; \kappa)^2$  as  $t \rightarrow -\infty$  can be calculated directly by the same method as the ones for  $t \rightarrow +\infty$ . We will not present this computation since it is mostly identical to the one in [21], and alternatively this asymptotics can be obtained using the connection formulae for the Painlevé II equation [26]. Moreover, the following singular asymptotics take place when  $\operatorname{Re} \beta = 1/2$ .

**Theorem 4.** Let  $u(t; \kappa)$  be the solution to the Painlevé II equation (1.8) characterized by (1.9) and let  $\kappa^2 = 1 - e^{-2i\pi\beta} = 1 + e^{2i\pi\gamma}$ ,  $\beta = 1/2 + i\gamma$ ,  $\gamma \in \mathbb{R}$ . Then  $y(t; \beta) = u(t; \kappa)^2$  is a solution to the Painlevé XXXIV equation (1.12) and has the following asymptotics as  $t \rightarrow -\infty$ , away from the zeros of trigonometric functions appearing in the denominators:

$$y\left(t; \frac{1}{2} + i\gamma\right) = \frac{-t}{\cos^2 \tilde{\phi}} + \frac{1}{\sqrt{-t}} \left( -\gamma + \frac{1}{2} \operatorname{tg} \tilde{\phi} + \frac{2\gamma}{\cos^2 \tilde{\phi}} + \frac{3(12\gamma^2 - 1) \sin \tilde{\phi}}{16 \cos^3 \tilde{\phi}} \right) + O\left(\frac{1}{t^2}\right), \quad (1.22)$$

$$\text{where } \tilde{\phi}(t; \gamma) = \frac{2}{3} (-t)^{3/2} + \frac{3}{2} \gamma \log(-t) + 3\gamma \log 2 - \arg \Gamma\left(\frac{1}{2} + i\gamma\right). \quad (1.23)$$

Asymptotics of this type in relation to the second Painlevé equation have been previously obtained via different methods in [26] and in [9], but the second term is a new result of the present work. This computation is based on an undressing procedure adapted from [9]. Thus we will not present the derivation of (1.22) either.

For the recurrence coefficients  $R_n$  and  $Q_n$ , we have the following result, which was partially obtained before in [33], see Remark 7 below.

**Theorem 5.** *Let  $R_n$  and  $Q_n$  be the recurrence coefficients defined in (1.4), associated to the orthogonal polynomials with respect to the weight (1.2). Let  $|\operatorname{Re} \beta| < 1/2$  and let  $\lambda_0$  be given by (1.7). Then, as  $n \rightarrow \infty$ , the recurrence coefficients have the following expansions,*

$$R_n(\lambda_0, \beta) = \frac{n}{2} - \frac{1}{2}u(t; \kappa)^2 n^{1/3} + O(1), \quad (1.24)$$

and

$$Q_n(\lambda_0, \beta) = -\frac{1}{\sqrt{2}}u(t; \kappa)^2 n^{-1/6} + O(n^{-1/2}), \quad (1.25)$$

uniformly for  $t \in [-M, \infty]$  for any  $M > 0$ , where  $\kappa$  is given by (1.16). Additionally, we have the asymptotics of the normalizing coefficients  $h_n$ :

$$h_n = \frac{\pi \sqrt{2nn^n}}{2^n e^n} e^{i\pi\beta} \left( 1 + n^{-1/3} v(t; \kappa) dt + n^{-2/3} \frac{1}{2} (v(t; \kappa)^2 - u(t; \kappa)^2) + O(n^{-1}) \right), \quad (1.26)$$

where

$$v(t; \kappa) = \int_t^\infty u(\tau; \kappa)^2 d\tau.$$

**Remark 6.** The formal substitution,

$$t = -2(1 - \lambda)n^{2/3}$$

in the asymptotics for the recurrence coefficients transforms them, with the help of the asymptotic expansion (1.10), into the non-critical asymptotics obtained in [21]. This important fact indicates, at least on the formal level, that the description of the transition regime in the large  $n$  behavior of the recurrence coefficients is complete.

**Remark 7.** The general form of (1.24) and (1.25) was formally suggested in [23] (together with the asymptotic characterization of the Painlevé II function  $u(t; \kappa)$ ) and it was proved by Xu and Zhao in [33]. They obtained their asymptotic expansions in terms of  $\hat{u}(t) = 2^{1/3}u(2^{-1/3}\tau)^2$ . It was noted that this is a solution of a Painlevé XXXIV equation, but no asymptotics for  $\hat{u}(t)$  as  $t \rightarrow \pm\infty$  were obtained, and thus the authors of [33] did not identify  $\hat{u}$  in terms of the Ablowitz-Segur solution characterized by (1.9) or (1.10). In fact, assuming the matching of the estimates (1.24) and (1.25) with the non-critical formulae of [21], asymptotics for  $\hat{u}(t)$  as  $t \rightarrow -\infty$  were deduced heuristically. There is, however, no independent derivation of it which is needed for the rigorous completion of the analysis of the transition regime in question. The  $+\infty$  - characterization of the Painlevé transcendent  $\hat{u}(t)$ , even heuristically, is not given in [33].

As an additional result, we also obtain an analog of the Plancherel-Rotach asymptotics for classical Hermite polynomials [29].

**Theorem 8.** *Let  $p_n(x)$  be the degree  $n$  monic orthogonal polynomial with respect to the weight (1.2), and let  $\lambda_0$  be given by (1.7). Let  $|\operatorname{Re} \beta| < 1/2$ . Then, as  $n \rightarrow \infty$ ,*

$$p_n(\lambda_0) = \frac{\sqrt{2\pi}}{\kappa} \left( \frac{ne}{2} \right)^{n/2} n^{1/6} e^{tn^{1/3}} u(t; \kappa) \left( 1 + O(n^{-1/3}) \right), \quad (1.27)$$

with  $\kappa$  given by (1.16).

*Remark 9.* Using the asymptotic behavior (1.9) for  $u$  as  $t \rightarrow +\infty$ , (1.27) matches formally with the classical Plancherel-Rotach asymptotics for the Hermite polynomials [29]:

$$p_n(\lambda_0) = \sqrt{2\pi} \left(\frac{ne}{2}\right)^{n/2} n^{1/6} e^{m^{1/3}} \text{Ai}(t) \left(1 + \mathcal{O}(n^{-1/3})\right), \quad n \rightarrow \infty, \quad (1.28)$$

where  $p_n$  are the monic Hermite polynomials.

On the other hand, if we let  $\beta \rightarrow 0$ , or equivalently  $\kappa \rightarrow 0$ , we have (see equations (5.29) and (5.31) below) that

$$u(t; \kappa) = 0, \quad \lim_{\kappa \rightarrow 0} \frac{1}{\kappa} u(t; \kappa) = \text{Ai}(t), \quad (1.29)$$

and this allows us to recover (1.28) also in this limit.

*Remark 10.* Consider the case of purely imaginary  $\beta$ , i.e.  $\beta = i\gamma$ ,  $\gamma \in \mathbb{R}$ . Then the three-term relation (1.4) generates in the usual way (see e.g. [11]) a symmetric on  $l_2$  Jacobi operator,  $L^0$ , defined by the semi-infinite matrix

$$L_{n,m}^0 = R_{n+1}^{1/2} \delta_{n+1,m} + Q_n \delta_{n,m} + R_n^{1/2} \delta_{n-1,m}, \quad n, m \geq 0, \quad R_0 = 0$$

whose domain is  $D = \{p = (p_0, p_1, \dots)^T \in l_2 : p_k = 0 \text{ for sufficiently large } k\}$ . Since the moment problem for the measure

$$d\mu(x) = w(x) dx$$

with the density  $w(x)$  given by (1.2) is determinate, the operator  $L^0$  is essentially self-adjoint and  $d\mu(x)$  is the spectral measure of its closure  $L \equiv \bar{L}^0$ . Therefore, the results of our last two theorems provide an insight into the properties of semi-infinite Jacobi matrices, i.e. discrete Schrödinger operators on a half-line, whose spectral densities have discontinuities. In the earlier works [10, 21, 25] it was demonstrated that the discontinuities in the spectral density are responsible for the oscillatory pattern in the large  $n$  asymptotics of the entries of the Jacobi matrix (in the coordinate asymptotics of the potentials of the discrete Schrödinger operator). More precisely, the oscillations occur when the point of the jump of the density is inside the support of the corresponding equilibrium measure. If the jump is outside, the behavior of the potentials  $R_n$  and  $Q_n$  is monotone. Formulae (1.24), (1.25) and (1.27) describe the corresponding transition regime. The formulae show that if the jump happens near the edge of the support then the large  $n$  (coordinate) asymptotics is governed by the Ablowitz-Segur solution of the second Painlevé equation and the parameters of the solution are explicitly related to the size of the jump. We actually believe that this fact is universal, i.e. the transitional formulae will be the same even if the Gaussian background in the spectral measure is replaced by an arbitrary exponential weight.

Our proofs of Theorem 5 and Theorem 8 are based on the nonlinear steepest descent method of Deift and Zhou (or, rather on its adaptation [15] to the Riemann-Hilbert (RH) problems related to the orthogonal polynomials [19]). This method was applied in [33] to the case of a discontinuous Gaussian weight with the point of discontinuity scaled as in (1.7). We will rely on the transformations and results from this paper, but we will adapt them in such a way that we can identify the function  $u(t; \kappa)$  as the Painlevé II solution with asymptotics (1.9) and (1.10). The RH analysis is presented in Section 3, and the proofs of Theorem 5 and Theorem 8 are given in Section 5.

Theorem 2 can be proved in two different ways. The first one is very short and relies on the Tracy-Widom formula (1.15) and on known asymptotic results in the Gaussian Unitary Ensemble. This proof will be given in Section 2. The second proof, given in Section 6, is lengthy but has the advantage of being self-contained. It relies on the RH analysis which we need anyways for the asymptotics of the orthogonal polynomials and their recurrence coefficients. As is always the case in the asymptotic analysis of Hankel and Toeplitz determinants,



the move from the asymptotics for the orthogonal polynomials and its recurrence coefficients to the asymptotics for the Hankel determinants is nontrivial. One has to address the “constant of integration problem” (c.f. [17]) which we do with the help of relevant differential identities for the Hankel determinant  $H_n(\lambda_0, \beta)$ .

In the RH analysis, we will identify the function  $u(t; \kappa)$  as the solution to the Painlevé II equation with asymptotics (1.9)–(1.10) using Lax pair arguments and an asymptotic analysis for a certain model RH problem (see Section 4), which is equivalent to the one which appeared in [33]. Solvability of this model RH problem was proved in [33], and we prove Theorem 1 as a consequence of this in Section 3.5.

The analysis in this paper shows similarities with the work [22] where a Painlevé XXXIV function appeared in a parametrix for a different type of critical edge behavior in unitary random matrix ensembles, namely with a root singularity instead of the jump singularity which we consider here. The RH problem which we study differs, however, from the one analyzed in [22]. This yields, in particular, serious technical differences in the analysis of the large positive  $t$  behavior of the Painlevé transcendent.

*Remark 11.* As it has already been indicated, it is Painlevé XXXIV equation (1.12) and the corresponding model RH problem that appear naturally during the asymptotic analysis of the orthogonal polynomials  $p_n(x)$ . The solution  $y(t; \beta)$  which emerges in this analysis is characterized by its RH data. We need to transform this characterization into the asymptotic behavior of  $y(t; \beta)$  as  $t \rightarrow \pm\infty$ . Because of the relation  $y = u^2$  between the solutions of Painlevé XXXIV equation (1.12) and the solutions of Painlevé II equation (1.8), one could think that the needed asymptotics could be extracted from the work of A. Kapaev [26], where the complete list of the global asymptotics of the second Painlevé transcendent is presented. However, to be able to use the results of [26] one needs to connect the RH data of  $y(t)$  with the RH data of  $u(t)$ . A well-known though still striking fact (see e.g. Chapter 5 of [20]) is that there is no simple relation between the Lax pair and the RH problem for the Painlevé XXXIV equation (1.12) and the standard Flaschka-Newell Lax pair (which is used in [26]) and the RH problem for the Painlevé II equation (1.8). Hence one does not know a priori the asymptotics of  $u(t)$ . There exists, however, a simple relation between the Lax pair and the RH problem for the Painlevé XXXIV equation (1.12) and the Lax pair and the RH problem for the nonuniform second Painlevé equation

$$q_{tt} = tq + 2q^3 - \frac{1}{2}, \quad (1.30)$$

so that one can use [26] and determine the asymptotics of  $q(t)$ . Unfortunately, now the problem with translation of the asymptotics of  $q(t)$  into the asymptotics of  $y(t)$  arises. The fact of the matter is that the relation between the Painlevé functions  $y(t)$  and  $q(t)$  is more complicated than the relation between the Painlevé functions  $y(t)$  and  $u(t)$ . Indeed, one has that

$$y(t) = 2^{-1/3} U(-2^{1/3} t), \quad U(t) = q^2(t) + q'(t) + \frac{t}{2} \quad (1.31)$$

(see e.g. [22, Appendix A]). This formula virtually destroys the asymptotic information which one could obtain for the function  $q(t)$  from [26]. For instance, one finds from [26] that the function  $q(t)$  behaves as  $\sim \sqrt{-t/2}$  as  $t \rightarrow -\infty$ . This, as we know *a posteriori*, must translate to the exponentially decaying asymptotics of  $y(t)$  as  $t \rightarrow +\infty$ . It is extremely difficult to verify this directly using (1.31): one has to prove cancellation of an asymptotic series in all orders of magnitude. Even worse is the situation with the asymptotics of  $q(t)$  as  $t \rightarrow +\infty$ . It is singular (and is described in terms of the cotangent function) and, after substitution into (1.31) it should transform into an oscillatory smooth decaying asymptotics. We refer the reader to [22], where a similar phenomenon had already been encountered, for more details. The above discussion makes it clear that, in spite of the simple relation to the second Painlevé function  $u(t)$ , an independent asymptotic analysis of the Painlevé XXXIV function  $y(t)$  is necessary. Of course, it is enough to evaluate the asymptotics of  $y(t; \beta)$  either for  $t \rightarrow +\infty$  or for  $t \rightarrow -\infty$ , since the one-end asymptotics will enable us to identify the function  $u(x)$  and use

[26] to determine its asymptotics on the another end. We have chosen to evaluate the asymptotics of  $y(t; \beta)$  as  $t \rightarrow +\infty$ . The relevant nonlinear steepest descent analysis is presented in Section 4. This analysis has some new technical features which are specifically indicated at the beginning of Section 4.

## 1.1 Applications

We conclude this introduction by indicating some applications of our results.

### 1.1.1 Random matrix moment generating function

Consider the  $n$ -dimensional Gaussian Unitary Ensemble (GUE) normalized such that the joint eigenvalue probability distribution is given by

$$\frac{1}{Z_n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{j=1}^n e^{-x_j^2} dx_j, \quad x_1, \dots, x_n \in \mathbb{R}. \quad (1.32)$$

The partition function  $Z_n$  is then equal to  $n! H_n(\lambda_0, 0)$ , with  $H_n(\lambda_0, 0)$  given in (1.6). For an  $n \times n$  GUE matrix, define the random variable  $X_{\lambda_0, n}$  as

$$X_{\lambda_0, n} = \text{number of eigenvalues greater than } \lambda_0. \quad (1.33)$$

It is natural to ask how the average of  $X_{\lambda_0, n}$  or its variance behaves for large  $n$ . The Hankel determinant with a discontinuous Gaussian weight carries information about such quantities. Indeed, the moment generating function of the random variable  $X_{\lambda_0, n}$ , which is defined as  $M_{\lambda_0, n}(y) := \mathbb{E}_n(e^{yX_{\lambda_0, n}})$ , can be expressed as

$$M_{\lambda_0, n}(y) = \frac{1}{Z_n} \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{j=1}^n \left( e^{-x_j^2} \times \begin{cases} 1, & x_j < \lambda_0 \\ e^y, & x_j \geq \lambda_0 \end{cases} \times dx_j \right). \quad (1.34)$$

This is in fact the ratio of two Hankel determinants, one with a discontinuous Gaussian weight, and one with a regular Gaussian weight: if we write  $y = -2\pi i\beta$ , we have

$$M_{\lambda_0, n}(y) = \frac{e^{-\pi i n \beta} H_n(\lambda_0, \beta)}{H_n(\lambda_0, 0)}. \quad (1.35)$$

This is true for any  $n$  and  $\lambda_0$ .

The large  $n$  asymptotics for the Hankel determinant  $H_n$  proved in Theorem 2 together with the explicit expression (1.6) for  $H_n(\lambda_0, 0)$ , immediately give information about the moment generating function as  $n \rightarrow \infty$  if  $\lambda_0$  is scaled as in (1.7).

Expanding the moment generating function for small values of  $y$ , we have

$$M_{\lambda_0, n}(y) = 1 + y \mathbb{E}_n(X_{\lambda_0, n}) + \frac{y^2}{2} \mathbb{E}_n(X_{\lambda_0, n}^2) + \mathcal{O}(y^3), \quad y \rightarrow 0, \quad (1.36)$$

so the average and variance of  $X_{\lambda_0, n}$  can be read off immediately from the small  $\beta$  asymptotics for the Hankel determinant.

In particular, differentiating (1.13) with respect to  $\beta$  and using (1.35) and (1.36), we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}_n(X_{\lambda_0, n}^k) = \frac{1}{(-2\pi i)^k} \frac{d^k}{d\beta^k} \left( \exp \left( - \int_t^\infty (\tau - t) u(\tau; \kappa)^2 d\tau \right) \right) \Big|_{\beta=0}, \quad (1.37)$$



with  $\kappa$  given by (1.16), which means that the large  $n$  limit of the moments of the random variable  $X_{\lambda_0, n}$  can be expressed in terms of the Ablowitz-Segur Painlevé II solutions  $u(\tau; \kappa)$  and its  $\kappa$ -derivatives evaluated at  $\kappa = 0$ . Note that this differentiation is justified since the asymptotics (1.13) are known to be uniform in a small neighborhood of  $\beta = 0$ . The first  $\kappa$ -derivative of  $u$  is the Airy function, by (1.29), and this implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}_n(X_{\lambda_0, n}) = \int_t^{+\infty} (\tau - t) \text{Ai}(\tau)^2 d\tau = \frac{1}{3} \left( 2t^2 \text{Ai}(t)^2 - \text{Ai}(t) \text{Ai}'(t) - 2t \text{Ai}'(t)^2 \right). \quad (1.38)$$

The same formula can be derived directly from  $\lim_{n \rightarrow \infty} \mathbb{E}_n(X_{\lambda_0, n}) = \int_t^{+\infty} \rho(\tau) d\tau$ , where  $\rho(t) = K_{\text{Ai}}(t, t) = \text{Ai}'(t)^2 - \text{Ai}''(t) \text{Ai}(t)$  is the density for the largest eigenvalue. Similarly, the behavior of higher moments can also be studied via just the correlation functions  $\rho_m(x_1, \dots, x_m) = \det \left( K_{\text{Ai}}(x_i, x_j) \right)_{i,j=1}^m$ . We would like to thank Peter Forrester for pointing out this fact.

### 1.1.2 Largest eigenvalue in a thinned GUE

The second application is connected to the so-called *thinning procedure* in the GUE. Consider the  $n$  eigenvalues  $x_1 \geq \dots \geq x_n$  of a GUE matrix, and apply the following thinning or filtering procedure to them: for each eigenvalue independently, we remove it with probability  $s \in (0, 1)$ . This leads us to a particle configuration, where the number of remaining particles can be any integer  $\ell$  between 0 and  $n$ , and we denote those particles by  $\mu_1 \geq \dots \geq \mu_\ell$ . Below, we show that the largest particle distribution in this process can be expressed in terms of a Hankel determinant with discontinuous Gaussian weight. More precisely, we have

$$\text{Prob}_s(\mu_1 \leq \lambda_0) = M_{\lambda_0, n}(\log s), \quad (1.39)$$

where  $M_{\lambda_0, n}(t)$  is defined in (1.35).

To prove (1.39), write  $E_n(k, \lambda_0)$  for the probability that a  $n \times n$  GUE matrix has exactly  $k$  eigenvalues bigger than  $\lambda_0$ . If we want none of the thinned or filtered particles  $\mu_1, \dots, \mu_\ell$  to be bigger than  $\lambda_0$ , that means that all GUE eigenvalues which are bigger than  $\lambda_0$  have to be removed by the thinning procedure. Therefore, we have

$$\text{Prob}_s(\mu_1 \leq \lambda_0) = \sum_{k=0}^n E_n(k, \lambda_0) s^k, \quad (1.40)$$

since each eigenvalue is removed independently with probability  $s$ .

Using the integral representation (1.34), it is on the other hand straightforward to show that

$$M_{\lambda_0, n}(\log s) = \sum_{k=0}^n E_n(k, \lambda_0) s^k. \quad (1.41)$$

Alternatively, this follows from the equation

$$E_n(k, \lambda_0) = \frac{1}{k!} \left( \frac{d}{ds} \right)^k M_{\lambda_0, n}(\log s), \quad (1.42)$$

which is well-known and proved, for example, in [28, Ch. 6 and 24]. Combining (1.40) with (1.41), we obtain (1.39). Consequently, by (1.35) and (1.13),

$$\lim_{n \rightarrow \infty} \text{Prob}_s(\mu_1 \leq \lambda_0) = \lim_{n \rightarrow \infty} M_{\lambda_0, n}(\log s) = \exp \left( - \int_t^{+\infty} (\tau - t) u(\tau; \kappa)^2 d\tau \right), \quad (1.43)$$

where  $s = 1 - \kappa^2$ . This relation, without the Hankel determinant, was discussed previously in [5, 6], where a transition was observed from the Tracy-Widom distribution (at  $s = 0$ ) to the Weibull distribution (at  $s = 1$ ). It is challenging, however, to describe explicitly the transition asymptotic regime from the behavior (1.17) corresponding to  $\beta = -\frac{i}{2\pi} \ln s$ ,  $0 < s \leq 1$  to the Tracy-Widom asymptotic behavior,

$$\ln \det \left( 1 - K_{\text{Ai}} \Big|_{[t, +\infty)} \right) = \frac{1}{12} t^3 - \frac{1}{8} \ln(-t) + \frac{1}{24} \ln 2 + \zeta'(-1) + o(1), \quad t \rightarrow -\infty, \quad (1.44)$$

corresponding to  $s = 0$ , i.e.  $\beta = -i\infty$  or  $\kappa = 1$ . Here,  $\zeta$  is the Riemann zeta-function. Similar transition regime for the sine-kernel determinant has been already described in [8] in terms of elliptic functions, and the presence of a very interesting cascade-type asymptotic behavior has been detected (see also [18] where the problem was analyzed, on a heuristic level, for the first time). In the case of the Airy-kernel, the question is still open, although on the level of the logarithmic derivatives, i.e. on the level of the Painlevé function  $u(t; \kappa)$ , the transition asymptotics from the Ablowitz-Segur case ( $\kappa < 1$ ) to the Hastings-McLeod ( $\kappa = 1$ ) case has already been found in [7].

### 1.1.3 Random partitions

The Airy kernel Fredholm determinant can be interpreted in terms of random partitions. The Plancherel measure on the set of partitions of  $N \in \mathbb{N}$  is a well-known measure which has its origin in representation theory. It can be defined in an elementary way by the following procedure. Take a permutation  $\sigma$  in  $S_N$  and define  $x_1$  as the maximal length of an increasing subsequence of  $\sigma$ . Next, we define  $x_2$  by requiring that  $x_1 + x_2$  is the maximal total length of two disjoint increasing subsequences of  $\sigma$ . We proceed in this way, and define  $x_k$  recursively by imposing that  $x_1 + \dots + x_k$  is the maximal total length of  $k$  disjoint increasing subsequences of  $\sigma$ , and we continue until  $x_1 + \dots + x_k = N$ . This procedure associates a partition  $x_1 \geq \dots \geq x_n$  of  $N$  to a permutation  $\sigma \in S_N$ . The uniform measure on  $S_N$  induces a measure on the set of partitions of  $N$ , which is the Plancherel measure.

We now take a random partition  $x_1 \geq \dots \geq x_n$  of  $N$  with respect to the Plancherel measure. Then, the particles  $N^{-1/6}(x_i - 2\sqrt{N})$  converge to the Airy process as  $N \rightarrow \infty$ , see e.g. [30]. Therefore, if we apply the filtering procedure which removes each component  $x_i$  of the partition independently with probability  $s$ , we obtain a new partition  $\mu_1 \geq \dots \geq \mu_m$  of a number  $\ell \leq N$ . Using similar arguments as in [30], it follows that

$$\lim_{N \rightarrow \infty} \text{Prob}_s \left( N^{-1/6}(\mu_1 - 2\sqrt{N}) \leq t \right) = \det \left( 1 - (1-s) K_{\text{Ai}} \Big|_{[t, +\infty)} \right). \quad (1.45)$$

Note that the conjectured integral identity (1.18) is of value in relation to (1.43) and (1.45).

We want to conclude this section by making the following general remark. From the point of view of the random matrix theory the examples considered in this section indicate that, in fact, it is the whole Ablowitz-Segur family of the Painlevé II transcendents that could appear in the theory and not only the Hastings-McLeod solution. Regarding the second appearance, it has already been known due to Bohigas et. al. [5, 6], however the first and the third examples are apparently new.

## 2 Theorem 2 and Conjecture 3

### 2.1 Proof of Theorem 2

Denote  $K_n$  for the GUE eigenvalue correlation kernel

$$K_n(x, y) = e^{-(x^2+y^2)/2} \sum_{k=0}^{n-1} H_k(x) H_k(y), \quad (2.1)$$

built out of normalized degree  $k$  Hermite polynomials  $H_k$ , orthonormal with respect to the weight  $e^{-x^2}$ . Define  $G_{\lambda_0,n}(\kappa)$  by

$$G_{\lambda_0,n}(\kappa) = M_{\lambda_0,n}(\log(1 - \kappa^2)) = \frac{1}{Z_n} \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{j=1}^n \left( e^{-x_j^2} \times \begin{cases} 1, & x_j < \lambda_0 \\ 1 - \kappa^2, & x_j \geq \lambda_0 \end{cases} \times dx_j \right). \quad (2.2)$$

By (1.35), we have

$$G_{\lambda_0,n}(\kappa) = M_{\lambda_0,n}(\log(1 - \kappa^2)) = \frac{e^{-\pi i n \beta} H_n(\lambda_0, \beta)}{H_n(\lambda_0, 0)}, \quad (2.3)$$

with  $\kappa^2 = 1 - e^{-2\pi i \beta}$ .

Similarly as in (1.41), we have

$$G_{\lambda_0,n}(\kappa) = M_{\lambda_0,n}(\log(1 - \kappa^2)) = \sum_{k=0}^n (1 - \kappa^2)^k E_n(k, \lambda_0) = \det \left( 1 - \kappa^2 K_n \Big|_{[\lambda_0, +\infty)} \right), \quad (2.4)$$

where  $K_{\text{Ai}} \Big|_{[t, +\infty)}$  is the integral operator with kernel  $K_n$  acting on  $[\lambda_0, +\infty)$ , and the determinant is the Fredholm determinant (the proof of the last equality in a more general setting can be found in [28, §23.3]).

Another well-known result is the convergence of the kernel  $K_n$  to the Airy kernel

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}(y) \text{Ai}'(x)}{x - y} \quad (2.5)$$

if  $x, y$  are scaled properly around  $\sqrt{2n}$ :

$$\frac{1}{\sqrt{2n}^{1/6}} K_n \left( \sqrt{2n} + \frac{u}{\sqrt{2n}^{1/6}}, \sqrt{2n} + \frac{v}{\sqrt{2n}^{1/6}} \right) = K_{\text{Ai}}(u, v) + e^{-c(|u|+|v|)} o(1), \quad (2.6)$$

uniformly for  $u, v > -M$ ,  $M > 0$ , for some  $c > 0$ . Using a slightly stronger version of this Airy kernel limit, as in [16], one shows the convergence of the associated Fredholm determinants: if we scale  $\lambda_0$  as in (1.7), we have

$$\lim_{n \rightarrow \infty} G_{\lambda_0,n}(\kappa) = \lim_{n \rightarrow \infty} \det \left( 1 - \kappa^2 K_n \Big|_{[\lambda_0, +\infty)} \right) = \det \left( 1 - \kappa^2 K_{\text{Ai}} \Big|_{[t, +\infty)} \right), \quad (2.7)$$

uniformly for  $t \in (-M, +\infty)$  for any  $M > 0$ , where  $K_{\text{Ai}} \Big|_{[t, +\infty)}$  is the integral operator with kernel  $K_{\text{Ai}}$  acting on  $L^2(t, +\infty)$ .

Using the Tracy-Widom formula (1.15) together with (2.3) and (2.7), we obtain

$$H_n(\lambda_0, \beta) = e^{\pi i n \beta} H_n(\lambda_0, 0) \exp \left( - \int_t^\infty (\tau - t) u(\tau; \beta)^2 d\tau \right) (1 + o(1)), \quad (2.8)$$

as  $n \rightarrow \infty$ . This proves (1.13).

## 2.2 Motivation of Conjecture 3

In the case where  $\lambda_0 = \lambda \sqrt{2n}$  with  $\lambda \in (-1, 1)$ , asymptotics for the Hankel determinants  $H_n(\lambda_0, \beta)$  were obtained in [21] and are given by (1.5). The dependence of the error term on  $\lambda$  was not made explicit in [21], but it can

be seen from their analysis that the error term in (1.5) gets worse if  $\lambda$  approaches  $\pm 1$ . We hope that by a careful inspection of the estimates in [21], one can strengthen the error term and obtain

$$H_n(\lambda \sqrt{2n}, \beta) = H_n(\lambda_0, 0) G(1 + \beta)G(1 - \beta)(1 - \lambda^2)^{-3\beta^2/2}(8n)^{-\beta^2} \times \\ \times \exp\left(2in\beta\left(\arcsin \lambda + \lambda \sqrt{1 - \lambda^2}\right)\right) \left(1 + O\left(\frac{1}{(n^{2/3}(1 - \lambda))^\gamma}\right)\right), \quad |\operatorname{Re} \beta| < \frac{1}{4}, \quad (2.9)$$

for some  $\gamma > 0$ . The error term must be uniform as  $\lambda \uparrow 1$  at a sufficiently slow rate such that  $n^{2/3}(1 - \lambda)$  is sufficiently large, say larger than some fixed  $M > 0$ .

We now take  $\lambda = 1 + tn^{-2/3}/2$  with  $-t > 2M$ . On the one hand, we can apply (2.9). Expanding the right-hand side of (2.9) for large  $n$ , we obtain, after a straightforward calculation,

$$\log H_n(\lambda_0, \beta) - \log H_n(\lambda_0, 0) - \pi i n \beta \\ = -\frac{4}{3}i\beta(-t)^{3/2} - \frac{3}{2}\beta^2 \log(-t) + \log(G(1 - \beta)G(1 + \beta)) - 3\beta^2 \log 2 + \epsilon_n(t), \quad (2.10)$$

where  $|\epsilon_n(t)| \leq c/|t|^\gamma + d|t|^{5/2}n^{-2/3}$  for some  $c, d, \gamma > 0$ , if  $n$  and  $-t$  are sufficiently large. We thank the referees for pointing out the  $n$ -dependence of this error term.

On the other hand, by (1.13), we have

$$\log H_n(\lambda_0, \beta) - \log H_n(\lambda_0, 0) - \pi i n \beta = \log \det\left(1 - \kappa^2 K_{\text{Ai}}\big|_{[t, +\infty)}\right) + o(1), \quad n \rightarrow \infty. \quad (2.11)$$

Comparing (2.10) with (2.11), we obtain

$$\log \det\left(1 - \kappa^2 K_{\text{Ai}}\big|_{[t, +\infty)}\right) \\ = -\frac{4}{3}i\beta(-t)^{3/2} - \frac{3}{2}\beta^2 \log(-t) + \log(G(1 - \beta)G(1 + \beta)) - 3\beta^2 \log 2 + \epsilon_n(t) + o(1), \quad (2.12)$$

as  $n \rightarrow \infty$ . Letting first  $n \rightarrow \infty$  and then  $t \rightarrow -\infty$ , we obtain (1.17). The total integral identity (1.18) now follows easily from (1.17) and (1.15).

### 3 RH analysis of orthogonal polynomials

#### 3.1 Overview of transformations

Following [19] (see also [11] and [24]), consider the RH problem for the matrix-valued function  $Y(z)$  analytic in both upper and lower open half-planes with the following jump condition on the real axis:

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \quad (3.1)$$

where  $Y_\pm(x)$  is the limit of  $Y(x)$  as  $z$  approaches  $x$  from the upper (+) or lower (-) half plane, and with  $w(x)$  given by (1.2).  $Y$  has the asymptotic condition

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{n\sigma_3} \text{ as } z \rightarrow \infty, \quad (3.2)$$

where  $\sigma_3$  is the third Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The explicit solution of this problem is

$$Y(z) = \begin{pmatrix} p_n(z) & (2\pi i)^{-1} \int_{-\infty}^{\infty} \frac{p_n(x)w(x)}{x-z} dx \\ -2\pi i h_{n-1}^{-1} p_{n-1}(z) & -h_{n-1}^{-1} \int_{-\infty}^{\infty} \frac{p_{n-1}(x)w(x)}{x-z} dx \end{pmatrix}, \quad (3.3)$$

where  $p_n$  and  $p_{n-1}$  are the monic orthogonal polynomials of degree  $n$  and  $n-1$  with respect to the weight  $w(x) = w(x; \lambda_0, \beta)$  defined in (1.2), and  $h_{n-1} = \int_{-\infty}^{+\infty} p_{n-1}(x)^2 w(x) dx$ .

This RH problem for  $Y$  has been studied asymptotically, for large  $n$  and with  $\lambda_0$  scaled as in (1.7), in [33]. We give an overview of the series of transformations constructed in this asymptotic analysis, but refer the reader to [33] for more details. Define the function  $T(z)$  as

$$T(z) = e^{-n\frac{l}{2}\sigma_3} (2n)^{-n\sigma_3/2} Y(\sqrt{2n} \cdot z) e^{n(\frac{l}{2}-g(z))\sigma_3}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3.4)$$

where

$$l = -1 - 2 \log 2, \quad g(z) = \int_{-1}^1 \log(z-s) \psi(s) ds, \quad z \in \mathbb{C} \setminus (-\infty, 1], \quad \psi(s) = \frac{2}{\pi} \sqrt{1-s^2}. \quad (3.5)$$

Here, the logarithm is in its principle branch with branch cut in the negative direction, and  $\psi(s) > 0$  on  $(-1, 1)$ . As usual, this  $g(z)$  satisfies certain variational relations:

$$g_+(z) + g_-(z) = 2z^2 + l, \quad z \in (-1, 1), \quad (3.6)$$

$$g_+(z) + g_-(z) < 2z^2 + l, \quad z \in \mathbb{R} \setminus [-1, 1]. \quad (3.7)$$

Additionally, its jump across the real line is described by

$$g_+(z) - g_-(z) = \begin{cases} 2\pi i, & z \leq -1, \\ 2\pi i \int_z^1 \psi(s) ds, & z \in [-1, 1], \\ 0, & z \geq 1. \end{cases} \quad (3.8)$$

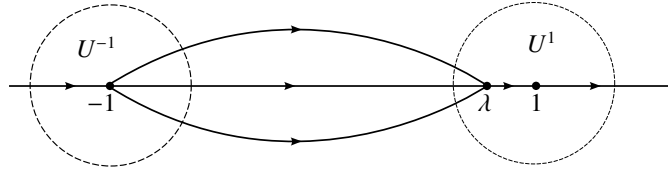
Let  $\psi(z)$  be the analytic continuation of  $\psi$  into  $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ . Introduce the function  $h(z)$  as follows:

$$h(z) = -\pi i \int_1^z \psi(y) dy, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)), \quad (3.9)$$

and define a piecewise analytic function  $S$  in lens-shaped regions (see Fig. 1) as follows:

$$S(z) = T(z) \cdot \begin{cases} I, & \text{outside the lenses,} \\ \begin{pmatrix} 1 & 0 \\ -e^{-i\pi\beta} e^{-2nh(z)} & 1 \end{pmatrix}, & \text{in the upper half-lens,} \\ \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} e^{2nh(z)} & 1 \end{pmatrix}, & \text{in the lower half-lens.} \end{cases} \quad (3.10)$$

As shown in [33], the function  $S$  has jumps on the lens-shaped contour shown in Fig. 1. As  $n \rightarrow \infty$ , the jump matrices tend to the identity matrix everywhere except on  $(-1, 1)$  and in small disks  $U^{-1}$  and  $U^1$  around  $-1$  and  $1$ . To obtain asymptotics for  $S$ , an outer parametrix and local parametrices near  $-1$  and  $+1$  have to be constructed.

Fig. 1: Opening of lens, case  $t < 0$ .

### 3.2 Outer parametrix

For  $z$  outside small disks around  $-1$  and  $+1$ ,  $S$  can be approximated for large  $n$  by an outer parametrix  $P^{(\infty)}$ , which is analytic except on  $[-1, 1]$ , tends to the identity as  $z \rightarrow \infty$ , and has the jump relation

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \cdot \begin{pmatrix} 0 & e^{\pi i \beta} \\ -e^{-\pi i \beta} & 0 \end{pmatrix}, \quad z \in (-1, 1). \quad (3.11)$$

It is given explicitly (see e.g. [21]) as

$$P^{(\infty)}(z) = \frac{1}{2} e^{i\pi\beta\sigma_3/2} \begin{pmatrix} a_0 + a_0^{-1} & -i(a_0 - a_0^{-1}) \\ i(a_0 - a_0^{-1}) & a_0 + a_0^{-1} \end{pmatrix} e^{-i\pi\beta\sigma_3/2}, \quad (3.12)$$

where

$$a_0(z) = \left( \frac{z-1}{z+1} \right)^{1/4}. \quad (3.13)$$

The branch of  $a_0$  is chosen so that  $a_0(z) \rightarrow 1$  as  $z \rightarrow \infty$ .

### 3.3 Local parametrix near 1

In order to obtain asymptotics for  $S$  also in neighborhoods of  $\pm 1$ , local parametrices have been constructed in [33]. Near  $-1$ , this local parametrix was built using the Airy function, but we do not need its explicit form. Near  $+1$ , it was built using a model RH problem associated to the Painlevé XXXIV equation.

The local parametrix  $P^{(1)}(z)$  is analytic in  $U^1$ , except for  $z$  on the jump contour for  $S$ , and it has the same jump relations as  $S$  for  $z$  on the jump contour for  $S$ , inside  $U^1$ . On the boundary  $\partial U^1$ , it satisfies the matching condition

$$P^{(1)}(z) \cdot P^{(\infty)}(z)^{-1} = I + O(n^{-1/3}) \text{ as } n \rightarrow \infty, \text{ uniformly for } z \in \partial U^1. \quad (3.14)$$

It takes the form

$$P^{(1)}(z) = E(z) \Phi(\zeta(z); \tau) e^{\frac{2}{3} \zeta(z)^{3/2} \sigma_3} e^{-i\pi\beta\sigma_3/2}, \quad (3.15)$$

where  $E$  is an analytic function in  $U^1$ ,  $\Phi$  will be specified below, and  $\zeta(z)$  is a conformal map near 1. The conformal map  $\zeta(z)$  and the parameter  $\tau$  are given by

$$\zeta(z) = \left( -\frac{3}{2} n h(z) \right)^{2/3}, \quad \tau = \zeta(\lambda) = \zeta \left( 1 + \frac{t}{2} n^{-2/3} \right). \quad (3.16)$$

Here, the multivalued power is the principle branch in  $\mathbb{C} \setminus (-\infty, 0)$ . The Taylor expansion at 1 is

$$\zeta(z) = 2n^{2/3} (z-1) \left( 1 + \frac{1}{10} (z-1) + O((z-1)^2) \right) \text{ as } z \rightarrow 1, \quad (3.17)$$

$$\tau = t + O(n^{-2/3}) \text{ as } n \rightarrow \infty. \quad (3.18)$$



The analytic pre-factor  $E$  can be expressed as

$$E(z) = P^{(\infty)}(z) e^{i\pi\beta\sigma_3/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \zeta(z)^{-\sigma_3/4}, \quad (3.19)$$

and  $\Phi(\zeta; \tau)$  is given by

$$\Psi_0(\zeta; \tau) = \begin{pmatrix} 1 & \frac{i\tau^2}{4} \\ 0 & 1 \end{pmatrix} \Phi(\zeta + \tau; \tau), \quad (3.20)$$

where  $\Psi_0(\xi; \tau)$  is the solution to the following RH problem.

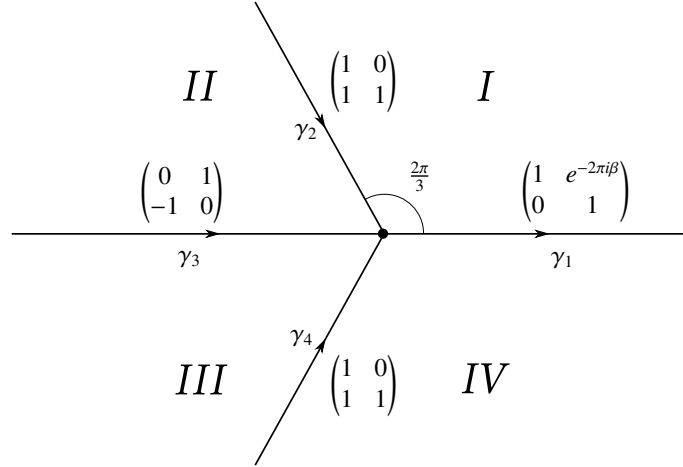


Fig. 2: The RH problem for  $\Psi_0(\xi)$ . The rays meet at  $\xi = 0$ . The union of the rays is referred to as  $\Gamma_{\Psi_0}$ .

$\Psi_0$  is analytic off the contour shown in Fig. 2 and satisfies the following jump and asymptotic conditions:

$$\Psi_{0+}(\xi) = \Psi_{0-}(\xi) \cdot \begin{cases} \begin{pmatrix} 1 & e^{-2\pi i \beta} \\ 0 & 1 \end{pmatrix}, & \xi \in \gamma_1, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \xi \in \gamma_2 \cup \gamma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \xi \in \gamma_3. \end{cases} \quad (3.21)$$

$$\Psi_0(\xi) = \left( I + \frac{m}{\xi} + O\left(\frac{1}{\xi^2}\right) \right) \xi^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-(\frac{2}{3}\xi^{3/2} + \tau\xi^{1/2})\sigma_3} \text{ as } \xi \rightarrow \infty, \quad (3.22)$$

$$\Psi_0(\xi) = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} + O(\xi) \right) \left( I + \frac{\kappa^2}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \log \xi \right) M(\xi) \text{ as } \xi \rightarrow 0, \quad (3.23)$$

where the matrix elements of  $m$  as well as  $a, b, c, d$  are some functions of  $\tau, \kappa$  is given by (1.16), and  $M$  is a

piecewise constant function defined as follows

$$M(\xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \xi \in I, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \xi \in II, \\ \begin{pmatrix} 1 - e^{-2\pi i \beta} & -e^{-2\pi i \beta} \\ 1 & 1 \end{pmatrix}, & \xi \in III, \\ \begin{pmatrix} 1 & -e^{-2\pi i \beta} \\ 0 & 1 \end{pmatrix}, & \xi \in IV. \end{cases} \quad (3.24)$$

All multivalued functions above are in their principle branches with branch cuts along the negative half axis.  $\Psi_0$  is uniquely determined by the above conditions. Note that all higher order terms in the expansions of  $\Psi_0$  are also functions of  $\tau$ .

The function  $P^{(1)}$  defined in (3.15) is the same as the one in [33], but it has to be noted that our functions  $\Psi_0$  is defined in a slightly different way compared to [33], which will be convenient later on. We have the relation

$$\Psi_0^{XZ}(\zeta; s) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} 2^{-\sigma_3/6} \Psi_0(\xi = 2^{2/3}\zeta; \tau = -2^{-1/3}s), \quad (3.25)$$

where  $\Psi_0^{XZ}$  denotes the solution to the model RH problem of [33].

By (3.20) and (3.22), it is straightforward to verify that  $\Phi$  admits the asymptotic expansion

$$\Phi(\zeta; \tau) = \left( I + \frac{m^\Phi}{\zeta} + O\left(\frac{1}{\zeta^2}\right) \right) \zeta^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-\frac{2}{3}\zeta^{3/2}\sigma_3} \text{ as } \zeta \rightarrow \infty, \quad (3.26)$$

where we have the following relation between  $m = m(\tau)$  and  $m^\Phi = m^\Phi(\tau)$ ,

$$m = \begin{pmatrix} m_{11}^\Phi + \frac{i\tau^2}{4}m_{21}^\Phi + \frac{\tau}{4} - \frac{\tau^4}{32} & m_{12}^\Phi - \frac{i\tau^2}{4}m_{11}^\Phi + \frac{\tau^4}{16}m_{21}^\Phi - \frac{i\tau^3}{12} + \frac{i\tau^6}{192} \\ m_{21}^\Phi + \frac{i\tau^2}{4} & m_{22}^\Phi - \frac{i\tau^2}{4}m_{21}^\Phi - \frac{\tau}{4} + \frac{\tau^4}{32} \end{pmatrix}. \quad (3.27)$$

### 3.4 Lax pair for $\Psi_0$ and the Painlevé XXXIV equation

From the RH conditions for  $\Psi_0$ , there is a standard procedure to deduce differential equations with respect to the variable  $\xi$  and the parameter  $\tau$ . Here, our approach deviates from the one in [33].

Consider the functions  $U := \frac{\partial \Psi_0}{\partial \xi} \Psi_0^{-1}$  and  $V = \frac{\partial \Psi_0}{\partial \tau} \Psi_0^{-1}$ . Because the jump matrices for  $\Psi_0$  are independent of  $\xi$  and  $\tau$ ,  $U$  and  $V$  are meromorphic functions of  $\xi$ . Using the behavior of  $\Psi_0$  at infinity and 0 given in (3.22) and (3.23), we obtain after a straightforward calculation that  $\Psi_0$  satisfies the Lax pair

$$\frac{\partial \Psi_0}{\partial \xi}(\xi; \tau) = U(\xi; \tau) \Psi_0(\xi; \tau), \quad U(\xi; \tau) = V(\xi; \tau) + \begin{pmatrix} 0 & -i\tau/2 \\ 0 & 0 \end{pmatrix} + \frac{\kappa^2}{2\pi i \xi} \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix}, \quad (3.28)$$

$$\frac{\partial \Psi_0}{\partial \tau}(\xi; \tau) = V(\xi; \tau) \Psi_0(\xi; \tau), \quad V(\xi; \tau) = -i\xi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} -m_{21} & 2m_{11} \\ -1 & m_{21} \end{pmatrix}, \quad (3.29)$$

where  $a, b, c, d$  and the matrix  $m$ , which are functions of the parameter  $\tau$  (and also of  $\beta$ ), were defined in (3.22)-(3.23). We can also refine the expansion for  $\frac{\partial \Psi_0}{\partial \tau} \Psi_0^{-1}$  as  $\xi \rightarrow \infty$ :

$$\frac{\partial \Psi_0}{\partial \tau} \Psi_0^{-1} - V(\xi, \tau) = \frac{1}{\xi} \frac{dm}{d\tau} - \frac{i}{\xi} \left[ m, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right] - \frac{i}{\xi} \left[ m, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] - \frac{i}{\xi} \left[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} m, m \right] + O\left(\frac{1}{\xi^2}\right). \quad (3.30)$$

Since this expression obviously has to be zero, equating its (21) entry to zero gives us the useful relation

$$m_{11} = \frac{1}{2} m_{21}^2 - \frac{i}{2} m'_{21}. \quad (3.31)$$

Note that  $m_{21}^2$  is the square of the matrix element  $m_{21}$ . The compatibility condition of the Lax system (3.28)-(3.29),

$$V_\xi - U_\tau = [U, V], \quad (3.32)$$

becomes

$$\begin{aligned} -i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} -m'_{21} & 2m'_{11} \\ 0 & m'_{21} \end{pmatrix} - \frac{\kappa^2}{2\pi i \xi} \begin{pmatrix} -(ac)' & (a^2)' \\ -(c^2)' & (ac)' \end{pmatrix} = \\ = \begin{pmatrix} \frac{\tau}{2} & -\tau m_{21} \\ 0 & -\frac{\tau}{2} \end{pmatrix} - i \frac{\kappa^2}{2\pi i} \begin{pmatrix} c^2 & -2ac \\ 0 & -c^2 \end{pmatrix} - \frac{i\kappa^2}{2\pi i \xi} \begin{pmatrix} 2c^2 m_{11} - a^2 & 2a^2 m_{21} - 4ac m_{11} \\ 2c^2 m_{21} - 2ac & a^2 - 2c^2 m_{11} \end{pmatrix}. \end{aligned} \quad (3.33)$$

This equation can be separated into two equations for each power of  $\xi$ . From the resulting system one can extract the equations

$$\frac{\kappa^2}{2\pi i} c^2 = m'_{21}(\tau) - \frac{i\tau}{2} = (m_{21}^\Phi)', \quad (3.34)$$

$$\frac{\kappa^2}{2\pi i} \left( ac - \frac{i\tau^2}{4} c^2 \right) = (m_{11}^\Phi)', \quad (3.35)$$

and

$$(1 + 2i\tau m_{21} - 4m'_{11})^2 + 4(2m'_{21} - i\tau)(2im'_{11} + 2im_{11}(\tau + 2im'_{21}) + \tau m'_{21} + m_{21}) = 0, \quad (3.36)$$

which, with the help of (3.31), reduces to

$$1 + 32\tau (m'_{21})^2 + 32i (m'_{21})^3 + 4im''_{21} - 4(m''_{21})^2 - 4i\tau m'''_{21} + 8m'_{21} (m'''_{21} - i\tau^2) = 0. \quad (3.37)$$

This equation is a disguised version of the 34th Painlevé equation for the function

$$y(\tau) = -im'_{21}(\tau) - \frac{\tau}{2} = -i(m_{21}^\Phi)'(\tau), \quad (3.38)$$

namely,

$$y_{\tau\tau} = 4y^2 + 2\tau y + \frac{(y_\tau)^2}{2y}. \quad (3.39)$$

Equation (3.34) also provides us with another representation of  $y(\tau)$ :

$$y(\tau) = i \lim_{\xi \rightarrow 0} \left[ \xi \frac{d\Psi_0(\xi)}{d\xi} \Psi_0^{-1}(\xi) \right]_{21}. \quad (3.40)$$

Moreover, from (3.31), we obtain an additional expression for  $y$  which does not involve derivatives:

$$y(\tau) = 2m_{11}(\tau) - m_{21}^2(\tau) - \tau/2. \quad (3.41)$$

### 3.5 Proof of Theorem 1

In [33, Corollary 1], it was proved using vanishing lemma techniques that the RH problem for  $\Psi_0^{XZ}(\zeta; s)$  is solvable for all real values of  $s$  if  $\beta$  is such that  $|\arg e^{-2i\pi\beta}| < \pi$ , and thus for all  $\beta$  such that  $|\operatorname{Re} \beta| < 1/2$ . Because of the explicit relation (3.25), this implies that the RH problem for  $\Psi_0$  is also solvable for all real values of  $\tau$  if  $|\operatorname{Re} \beta| < 1/2$ . This in turn implies that the function  $y(\tau) = y(\tau; \beta)$  defined in terms of  $\Psi_0$  by (3.41) is well-defined and cannot have singularities for real  $\tau$  if  $|\operatorname{Re} \beta| < 1/2$ .

If we define  $u(\tau; \kappa)$  by  $u(\tau; \kappa)^2 = y(\tau; \beta)$  with  $\kappa^2 = 1 - e^{-2\pi i \beta}$ , then it is easily verified by (3.39) that  $u$  solves the Painlevé II equation (1.8). By exploring the asymptotic behavior of  $y(\tau; \beta)$  (or, equivalently, of  $u(\tau; \kappa)$ ) as  $\tau \rightarrow \pm\infty$ , we will be able to identify  $u(\tau; \kappa)$  as the Ablowitz-Segur solution of the Painlevé II equation characterized by (1.9) and (1.10). This identification, which will follow from (4.33) below, completes the proof of Theorem 1.

### 3.6 Final transformation

Introduce the new function

$$R(z) = S(z) \cdot \begin{cases} \left(P^{(\infty)}(z)\right)^{-1}, & z \in \mathbb{C} \setminus \overline{U^{-1} \cup U^1 \cup (-1, 1)}, \\ \left(P^{(-1)}(z)\right)^{-1}, & z \in U^{(-1)}, \\ \left(P^{(1)}(z)\right)^{-1}, & z \in U^{(1)}, \end{cases} \quad (3.42)$$

which tends to the identity matrix as  $z \rightarrow \infty$  and which has jump matrices  $G_R$  on the contour  $\Gamma_R$  that tend to identity as  $n \rightarrow \infty$ :

$$G_R(z) = \begin{cases} P_+^{(\infty)}(z) \left(P_-^{(\infty)}(z)\right)^{-1}, & z \in (-1, 1) \setminus \overline{U^{-1} \cup U^1}, \\ P^{(\infty)}(z) \left(P^{(-1)}(z)\right)^{-1}, & z \in \partial U^{-1}, \\ P^{(\infty)}(z) \left(P^{(1)}(z)\right)^{-1}, & z \in \partial U^1, \end{cases} = I + O\left(\frac{1}{n^{1/3}(1+|z|)^p}\right), \quad z \in \Gamma_R. \quad (3.43)$$

This, in turn, implies (see [13]) that for sufficiently large  $n$

$$R(z) = I + O\left(\frac{1}{(1+|z|)n^{1/3}}\right), \quad \text{uniformly for } z \in \mathbb{C} \setminus \Gamma_R, \quad (3.44)$$

where  $\Gamma_R$  is the jump contour for  $R$ .

## 4 Asymptotics of $u(\tau; \kappa)$ as $\tau \rightarrow +\infty$

From Section 3.4, we know that  $y(\tau; \beta)$  defined by (3.38) solves the Painlevé XXXIV equation (3.39), and this implies that  $u$  defined by  $u(\tau; \kappa)^2 = y(\tau; \beta)$  (with relation (1.16) between  $\kappa$  and  $\beta$ ) solves the Painlevé II equation (1.8). We now proceed with proving the asymptotics of  $y(\tau; \beta) = u(\tau; \kappa)^2$  as stated in (1.9). In this section it is assumed that  $\tau > 0$ . The analysis performed here is largely analogous to the one in [22] with one additional new technical feature – the need to introduce an additional triangular parametrix near the point  $z = 0$  of the discontinuity of the (triangular) jump matrix (see Subsection 4.3.2).

## 4.1 Rescaling and shift of the jump contour

Introduce

$$A(z) = \tau^{-\sigma_3/4} \Psi_0(\tau z; \tau). \quad (4.1)$$

One can easily check that it satisfies the following RH problem.

- (a)  $A : \mathbb{C} \setminus \Gamma_{\Psi_0} \rightarrow \mathbb{C}^{2 \times 2}$  is analytic.
- (b)  $A$  has the same jump relations as  $\Psi_0$ .
- (c)  $A(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-\tau^{3/2}(\frac{2}{3}z^{3/2} + z^{1/2})\sigma_3}$  as  $z \rightarrow \infty$ .
- (d)  $A(z)$  inherits its behavior at  $z = 0$  from  $\Psi_0$  very easily.

From (3.40) we get

$$y(\tau) = \frac{i}{\sqrt{\tau}} \lim_{z \rightarrow 0} \left[ z \frac{dA(z)}{dz} A(z)^{-1} \right]_{21}. \quad (4.2)$$

We shall further write

$$s = \tau^{3/2}. \quad (4.3)$$

With respect to the domains defined in Fig. 3, define

$$B(z) = \begin{cases} A(z) \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in II', \\ A(z) \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & z \in III', \\ A(z), & z \in I \cup II'' \cup III'' \cup IV. \end{cases} \quad (4.4)$$

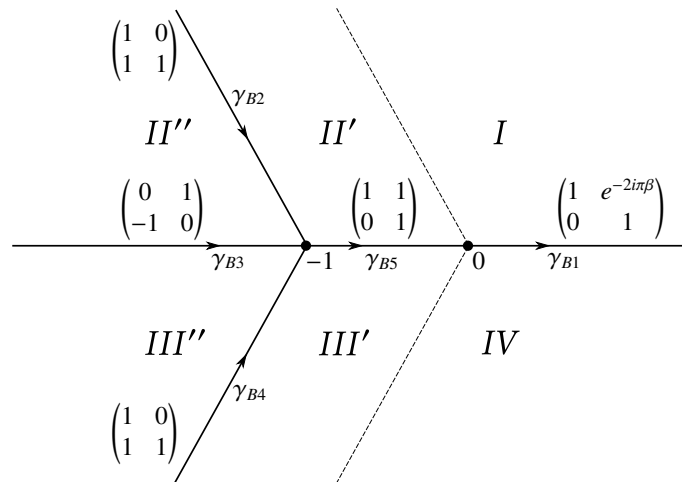


Fig. 3: The contours  $\Gamma_B$  and the RH problem for  $B(z)$ .

This function satisfies the following RH problem.

- (a)  $B : \mathbb{C} \setminus \Gamma_B \rightarrow \mathbb{C}^{2 \times 2}$  is analytic.

$$(b) \quad B_+(z) = B_-(z) \cdot \begin{cases} \begin{pmatrix} 1 & e^{-2\pi i \beta} \\ 0 & 1 \end{pmatrix}, & z \in \gamma_{B1}, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \gamma_{B2} \cup \gamma_{B4}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{B3}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \gamma_{B5}. \end{cases}$$

$$(c) \quad B(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-s\left(\frac{2}{3}z^{3/2} + z^{1/2}\right)\sigma_3} \text{ as } z \rightarrow \infty.$$

(d)  $B(z)$  has logarithmic behavior near  $z = 0$ . Namely,

$$B(z) = \tilde{B}(z) \left( I + \frac{\kappa^2}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \log z \right) M_{\pm}, \quad z \in \mathbb{H}^{\pm}, \quad (4.5)$$

where  $\tilde{B}(z)$  is some analytic function,  $M_+ = I$  and  $M_- = \begin{pmatrix} 1 & -e^{-2\pi i \beta} \\ 0 & 1 \end{pmatrix}$ . The logarithm is in its principle branch with branch cut along the negative half axis.

Finally, the expression for  $y(\tau)$  remains unchanged compared to (4.2),

$$y(\tau) = \frac{i}{\sqrt{\tau}} \lim_{z \rightarrow 0} \left[ z \frac{dB(z)}{dz} B(z)^{-1} \right]_{21}. \quad (4.6)$$

This transformation is an important precursor of the introduction of a new  $g$ -function that will allow us to “undress” the behavior of the RH problem at infinity (for similar transitions see e.g. [22] and [32]).

## 4.2 Normalization at infinity

We now introduce the following  $g$ -function,

$$\hat{g}(z) = \frac{2}{3}(z+1)^{3/2}, \quad -\pi < \arg(z+1) < \pi. \quad (4.7)$$

Note that

$$\hat{g}(z) - \left( \frac{2}{3}z^{3/2} + z^{1/2} \right) = \frac{1}{4z^{1/2}} + O\left(\frac{1}{z^{3/2}}\right) \text{ as } z \rightarrow \infty, \quad z \notin (-\infty, -1]. \quad (4.8)$$

Next, define

$$C(z) = \begin{pmatrix} 1 & -is/4 \\ 0 & 1 \end{pmatrix} B(z) e^{s\hat{g}(z)\sigma_3}. \quad (4.9)$$

The constant prefactor in this definition is needed to conserve the  $O\left(\frac{1}{z}\right)$  term in the asymptotics as  $z \rightarrow \infty$ .  $C$  satisfies the following RH problem.

(a)  $C : \mathbb{C} \setminus \Gamma_B \rightarrow \mathbb{C}^{2 \times 2}$  is analytic.



- (b)  $C_+(z) = C_-(z) \cdot \begin{cases} \begin{pmatrix} 1 & e^{-2\pi i \beta} e^{-2s\hat{g}(z)} \\ 0 & 1 \end{pmatrix}, & z \in \gamma_{B1}, \\ \begin{pmatrix} 1 & 0 \\ e^{2s\hat{g}(z)} & 1 \end{pmatrix}, & z \in \gamma_{B2} \cup \gamma_{B4}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_{B3}, \\ \begin{pmatrix} 1 & e^{-2s\hat{g}(z)} \\ 0 & 1 \end{pmatrix}, & z \in \gamma_{B5}. \end{cases}$
- (c)  $C(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$  as  $z \rightarrow \infty$ ,  $z \notin \Gamma_B$ .
- (d)  $C(z) = \tilde{C}(z) \left(I + \frac{\kappa^2}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \log z\right) M_{\pm} e^{s\hat{g}(z)\sigma_3}$ , for  $z \in \mathbb{C}^{\pm}$  near 0, where  $\tilde{C}$  is analytic in a neighborhood of 0 and  $M_{\pm}$  are the same as before. The branch of the logarithm is chosen as before.

From the definition of  $C$ ,

$$\left[ z \frac{dC(z)}{dz} C(z)^{-1} \right]_{21} = \left[ z \frac{dB(z)}{dz} B(z)^{-1} \right]_{21} + s\hat{g}'(z) \left[ zB(z)\sigma_3 B(z)^{-1} \right]_{21}. \quad (4.10)$$

The second term in this expression tends to zero as  $z \rightarrow 0$  due to the behavior of  $B(z)$ , hence

$$y(\tau) = \frac{i}{\sqrt{\tau}} \lim_{z \rightarrow 0} \left[ z \frac{dC(z)}{dz} C(z)^{-1} \right]_{21}. \quad (4.11)$$

### 4.3 Construction of parametrices

#### 4.3.1 Global Airy solution $C^{(\text{Ai})}$

The jumps of  $C(z)$  near  $z = -1$  are very similar to the jumps of the standard Airy RH problem. Let us look for a function  $C^{(\text{Ai})}$  that satisfies the following RH problem.

- (a)  $C^{(\text{Ai})} : \mathbb{C} \setminus \Gamma_B \rightarrow \mathbb{C}^{2 \times 2}$  is analytic.
- (b)  $C^{(\text{Ai})}(z)$  has the same jumps on  $\Gamma_B \setminus [-1, +\infty)$  as  $C(z)$  and its jump on  $(-1, +\infty)$  is  $\begin{pmatrix} 1 & e^{-2s\hat{g}(z)} \\ 0 & 1 \end{pmatrix}$ .
- (c)  $C^{(\text{Ai})}(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$  as  $z \rightarrow \infty$ .

We seek  $C^{(\text{Ai})}$  in the form

$$C^{(\text{Ai})}(z) = \hat{C}^{(\text{Ai})}(z) e^{s\hat{g}(z)\sigma_3}. \quad (4.12)$$

If we define an auxiliary matrix function (whose jumps are shown in Fig. 4)

$$\begin{cases} \Phi^{(\text{Ai})} = \begin{pmatrix} -y_1 & -y_2 \\ -y'_1 & -y'_2 \end{pmatrix} \text{ in } II, \\ \Phi^{(\text{Ai})} = \begin{pmatrix} -y_2 & y_1 \\ -y'_2 & y'_1 \end{pmatrix} \text{ in } III, \end{cases} \quad \begin{cases} \Phi^{(\text{Ai})} = \begin{pmatrix} y_0 & -y_2 \\ y'_0 & -y'_2 \end{pmatrix} \text{ in } I, \\ \Phi^{(\text{Ai})} = \begin{pmatrix} y_0 & y_1 \\ y'_0 & y'_1 \end{pmatrix} \text{ in } IV, \end{cases} \quad (4.13)$$

where

$$y_0(z) = \text{Ai}(z), \quad y_1(z) = e^{2\pi i/3} \text{Ai}(e^{2\pi i/3} z), \quad y_2(z) = e^{4\pi i/3} \text{Ai}(e^{4\pi i/3} z), \quad (4.14)$$

then a standard argument shows that  $\hat{C}^{(\text{Ai})}$  must have the form

$$\hat{C}^{(\text{Ai})}(z) = \sqrt{2\pi} \begin{pmatrix} 0 & -1 \\ -i & 0 \end{pmatrix} \tau^{\sigma_3/4} \Phi^{(\text{Ai})}(\tau(z+1)), \quad (4.15)$$

This in particular implies a refined asymptotics for  $C^{(\text{Ai})}$ :

$$C^{(\text{Ai})}(z) = \left( I + \frac{m^{\text{Ai}}}{z} + O\left(\frac{1}{z^2}\right) \right) z^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad (4.16)$$

$$m^{\text{Ai}} = \frac{\sigma_3}{4} + \frac{7i}{48s} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (4.17)$$

Airy solutions like this are discussed in much detail, for example, in [14].

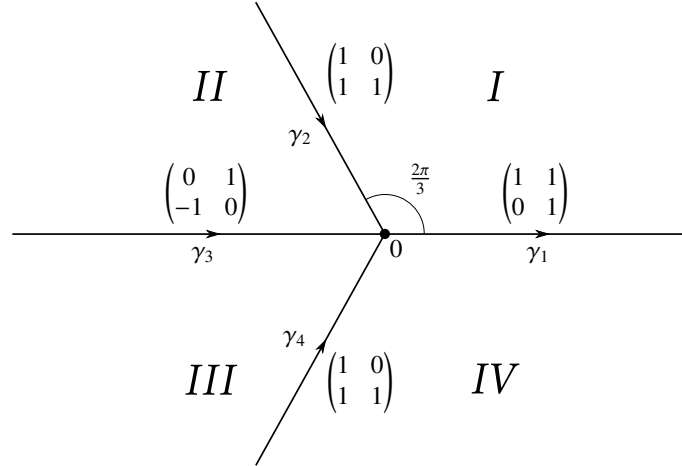


Fig. 4: The standard Airy RH problem. The union of contours is referred to as  $\Gamma_{\text{Ai}}$ .

#### 4.3.2 Local solution $C^{(0)}$

We also need a local parametrix for  $C$  near  $z = 0$ . Let  $U^0$  be a small open disk around 0 of radius less than 1, say,  $1/2$ . Then we have to find the function  $C^{(0)}$  which satisfies the following RH problem.

(a)  $C^{(0)} : \overline{U^0} \setminus [0, +\infty) \rightarrow \mathbb{C}^{2 \times 2}$  is analytic.

(b)  $C_+^{(0)}(z) = C_-^{(0)}(z) \begin{pmatrix} 1 & (e^{-2\pi i\beta} - 1)e^{-2s\hat{g}(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in (0, +\infty) \cap U^0$  (contour oriented to the right).

(c)  $C^{(0)}(z) = I + o(1)$  as  $s \rightarrow \infty$ , uniformly on  $\partial U^0$ .

(d)  $C^{(0)}(z) \sim \left( I + \frac{\kappa^2}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \log z \right) e^{s\hat{g}(z)\sigma_3}$  as  $z \rightarrow 0$ . The branch cut of the logarithm is along the positive half axis.

Due to the simple algebraic structure of the jumps, this problem can be solved exactly in terms of integrals of elementary functions. Namely, the solution is

$$C^{(0)}(z) = \begin{pmatrix} 1 & -\frac{\kappa^2}{2\pi i} \int_0^{1/2} \frac{e^{-2s\hat{g}(z')}}{z' - z} dz' \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C} \setminus \left[0, \frac{1}{2}\right]. \quad (4.18)$$

This function clearly has the requested jumps and has the same general logarithmic behavior near  $z = 0$ . Moreover, this function satisfies the matching condition on  $\partial U^0$  and, in fact, with some  $c > 0$  we have

$$C^{(0)}(z) = I + O(e^{-cs}) \text{ as } s \rightarrow \infty, \text{ uniformly on } \partial U^0. \quad (4.19)$$

We will also need the fact that

$$\lim_{z \rightarrow 0} z \frac{dC^{(0)}}{dz} (C^{(0)})^{-1} = \begin{pmatrix} 0 & \frac{\kappa^2}{2\pi i} e^{-2s\hat{g}(0)} \\ 0 & 0 \end{pmatrix}. \quad (4.20)$$

#### 4.4 Final transformation

Using the functions built in the previous subsection, we can now perform the final transformation of the RH analysis in the case where  $\tau \rightarrow +\infty$ .

Define

$$D(z) = \begin{cases} C(z) \cdot (C^{(0)}(z))^{-1} \cdot (C^{(Ai)}(z))^{-1}, & z \in U^0 \setminus \mathbb{R}, \\ C(z) \cdot (C^{(Ai)}(z))^{-1}, & z \in \mathbb{C} \setminus \overline{U^0 \cup \Gamma_B}. \end{cases} \quad (4.21)$$

This function has the following properties.

(a)  $D : \mathbb{C} \setminus \left(\left[\frac{1}{2}, +\infty\right) \cup \partial U^0\right) \rightarrow \mathbb{C}^{2 \times 2}$  is analytic.

(b) Assuming the counterclockwise orientation of  $\partial U^0$ ,

$$D_+(z) = D_-(z) \cdot \begin{cases} C_-^{(Ai)}(z) \cdot \begin{pmatrix} 1 & -\kappa^2 e^{-2s\hat{g}(z)} \\ 0 & 1 \end{pmatrix} \cdot (C_-^{(Ai)}(z))^{-1}, & z \in \left(\frac{1}{2}, +\infty\right), \\ C^{(Ai)}(z) \cdot (C^{(0)}(z))^{-1} \cdot (C^{(Ai)}(z))^{-1}, & z \in \partial U^0. \end{cases} \quad (4.22)$$

(c)  $D(z) = I + O\left(\frac{1}{z}\right)$ , as  $z \rightarrow \infty$ .

Using the asymptotic expansion for  $C^{(Ai)}$ , it is easy to check that, with some  $c > 0$ ,

$$(D_-(z))^{-1} D_+(z) = I + O(e^{-cs|z|}) \text{ as } s \rightarrow \infty, \text{ uniformly for } z \in \left(\frac{1}{2}, +\infty\right). \quad (4.23)$$

As for the jump on  $\partial U^0$ , by virtue of (4.19) and the boundedness of  $C^{(Ai)}$ , it is also close to the identity matrix:

$$(D_-(z))^{-1} D_+(z) = C^{(Ai)}(z) \cdot (I + O(e^{-cs})) \cdot (C^{(Ai)}(z))^{-1} = I + O(e^{-cs}) \quad (4.24)$$

as  $s \rightarrow \infty$ , uniformly for  $z \in \partial U^0$ , with some  $c > 0$ .

Using these estimates, in a standard way [13] one shows that, for any  $z \in \mathbb{C} \setminus \Gamma_D$ ,

$$D(z) = I + O\left(\frac{e^{-cs}}{1 + |z|}\right) \text{ as } s \rightarrow \infty, \quad c > 0. \quad (4.25)$$

The error term is uniform on compact subsets of  $\mathbb{C} \setminus \Gamma_D$ .

## 4.5 Asymptotics for $y$ and uniformity of error terms

Following the transformations  $\Phi \mapsto \Psi_0 \mapsto A \mapsto B \mapsto C \mapsto D$  (eqs. (3.20), (4.1), (4.4), (4.9), (4.21)) backwards, we can recover the connection between the asymptotic expansions of  $\Phi$  and  $D$ . Namely, for large  $z \in \mathbb{C} \setminus (II' \cup III')$ , we have

$$\Phi(z) = \tau^{\sigma_3/4} D\left(\frac{z}{\tau} - 1\right) \sqrt{2\pi} \begin{pmatrix} 0 & -1 \\ -i & 0 \end{pmatrix} \tau^{\sigma_3/4} \Phi^{(\text{Ai})}(z). \quad (4.26)$$

Next, we write, as usual,

$$D(z) = I + \frac{m^D}{z} + \frac{m^{D;2}}{z^2} + O\left(\frac{1}{z^3}\right) \text{ as } z \rightarrow \infty, \quad (4.27)$$

$$\Phi(z) = \left( I + \frac{m^\Phi}{z} + \frac{m^{\Phi;2}}{z^2} + O\left(\frac{1}{z^3}\right) \right) z^{\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-\frac{2}{3}z^{3/2}\sigma_3} \text{ as } z \rightarrow \infty, \quad (4.28)$$

Let us refer to the matrix coefficients in front of  $z^{-k}$  in these expansions as  $m^{D;k}$  and  $m^{\Phi;k}$  (they are only functions of  $\tau$ ). Using (4.26), we see that each of the matrices  $m^{\Phi;k}$  in the expansion for  $\Phi$  is merely a linear combination of a finite number of the matrix coefficients  $m^{D;k}$ , with coefficients rational in  $\tau^{1/4}$ .

Using (4.25), one shows that the matrices  $m^{D;k}$  are exponentially small,

$$m^{D;k}(\tau) = O\left(e^{-c\tau^{3/2}}\right) \text{ as } \tau \rightarrow +\infty, \text{ with some } c > 0 \quad (4.29)$$

for all  $k$ . It immediately follows that

$$m^{\Phi;k}(\tau) = \tilde{m}^{\text{Ai};k} + O\left(\tau^{k+\frac{1}{2}} e^{-c\tau^{3/2}}\right) \text{ as } \tau \rightarrow +\infty, \quad (4.30)$$

thus  $m^{\Phi;k}$  are bounded at large  $\tau$ .

These facts imply that the asymptotic expansion (3.26) for  $\Phi$  is *uniform* for  $\tau \in [\tau_0, +\infty)$  for any  $\tau_0 \in \mathbb{R}$ .

Since  $C(z) = D(z) \cdot C^{(\text{Ai})}(z) \cdot C^{(0)}(z)$  in  $U^0$  and both  $D(z)$  and  $C^{(\text{Ai})}$  are bounded there, we get from (4.11) that

$$y(\tau) = \frac{i}{\sqrt{\tau}} \lim_{z \rightarrow 0} \left[ z \frac{dC(z)}{dz} C(z)^{-1} \right]_{21} = \frac{\kappa^2 e^{-2s\hat{g}(0)}}{2\pi \sqrt{\tau}} \left[ D(0) C^{(\text{Ai})}(0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (D(0) C^{(\text{Ai})}(0))^{-1} \right]_{21} \text{ for large } \tau > 0. \quad (4.31)$$

From (4.12), (4.15), and the asymptotics for  $\Phi^{(\text{Ai})}$  it follows that

$$C^{(\text{Ai})}(0) = \sqrt{2\pi} \begin{pmatrix} 0 & -1 \\ -i & 0 \end{pmatrix} \tau^{\sigma_3/4} \Phi^{(\text{Ai})}(\tau) e^{\frac{2}{3}\tau^{3/2}\sigma_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} + O\left(\frac{1}{\tau^{3/2}}\right) \text{ as } \tau \rightarrow +\infty. \quad (4.32)$$

Taking into account that  $D(0)$  converges to  $I$  very rapidly and using the definitions (4.7) and (4.3), we arrive at the final expression for the asymptotic behavior of  $y(\tau; \beta)$ , where we emphasize the dependence on  $\beta$ :

$$y(\tau; \beta) = e^{-\frac{4}{3}\tau^{3/2}} \left( \frac{\kappa^2}{4\pi \sqrt{\tau}} + O\left(\frac{1}{\tau^2}\right) \right) = (\kappa \text{Ai}(\tau))^2 \left( 1 + O\left(\tau^{-2/3}\right) \right) \text{ as } \tau \rightarrow +\infty. \quad (4.33)$$

Together with  $y(\tau; \beta) = u(\tau; \kappa)^2$ , this proves (1.9).

## 5 Asymptotics of the recurrence coefficients

In this section, we compute asymptotics for the recurrence coefficients  $R_n$  and  $Q_n$ . Our calculations in this section are similar to those in [33], but we believe it is convenient for the reader to give some details of the calculations because of differences in notations.

## 5.1 Auxiliary asymptotics of $G_R$

We now need to compute the precise asymptotic behavior of  $G_R$ , the jump matrix for  $R$  (see (3.43)). Finding an explicit expression for the two leading terms in  $G_R$  on  $\partial U^1$  is the most sophisticated part of this calculation. First, expand

$$G_R(z) = P^{(\infty)}(z) \left( P^{(1)}(z) \right)^{-1} = P^{(\infty)}(z) e^{-\frac{2}{3}\zeta(z)^{3/2}\sigma_3} e^{i\pi\beta\sigma_3/2} \Phi(\zeta(z))^{-1} E(z)^{-1} \text{ for } z \in \partial U^1 \quad (5.1)$$

with  $E(z)$  defined in (3.19). Recall that we have the asymptotic expansion (3.26) for  $\Phi$ , uniformly for  $\tau \geq \tau_0$  with any fixed  $\tau_0 \in \mathbb{R}$ . Therefore, one verifies using (3.12) that, as  $n \rightarrow \infty$ ,

$$G_R(z) = e^{i\pi\beta\sigma_3/2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( \frac{z-1}{z+1} \right)^{\sigma_3/4} \left( I - \frac{m_{21}^\Phi}{\zeta(z)^{1/2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{m_{11}^\Phi}{\zeta(z)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + O(n^{-1}) \right) \times \\ \times \left( \frac{z-1}{z+1} \right)^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-i\pi\beta\sigma_3/2}, \quad (5.2)$$

which gives us the following expansion of  $G_R$  as  $n \rightarrow \infty$ :

$$G_R(z) = I - G_1(z)n^{-1/3} + G_2(z)n^{-2/3} + O(n^{-1}), \text{ uniformly for } z \in \partial U^1 \text{ and } \tau \geq \tau_0, \quad (5.3)$$

where

$$G_1(z) = \frac{im_{21}^\Phi (z+1)^{1/2}}{2} \frac{n^{1/3}}{(z-1)^{1/2} \zeta(z)^{1/2}} \begin{pmatrix} 1 & -ie^{i\pi\beta} \\ -ie^{-i\pi\beta} & -1 \end{pmatrix} \quad (5.4)$$

and

$$G_2(z) = \frac{im_{11}^\Phi n^{2/3}}{\zeta(z)} \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}. \quad (5.5)$$

## 5.2 Asymptotics of $R_n$

We can use the following simple identity for the recurrence coefficient  $R_n$  defined in (1.4) (see e.g. [11]):

$$R_n = m_{12}^Y m_{21}^Y, \quad (5.6)$$

where the matrix  $m^Y$  is defined in terms of the large  $z$  expansion of  $Y$ :

$$Y(z) = \left( I + \frac{m^Y(t)}{z} + O\left(\frac{1}{z^2}\right) \right) z^{n\sigma_3}. \quad (5.7)$$

In order to compute  $m^Y$ , we will need similar large  $z$  expansions for the following functions

$$R(z) = I + \frac{m^R(t)}{z} + O\left(\frac{1}{z^2}\right), \quad (5.8)$$

$$P^{(\infty)} = I + \frac{m^\infty(t)}{z} + O\left(\frac{1}{z^2}\right), \quad (5.9)$$

$$g(z) = \log z - \frac{1}{8z^2} + O\left(\frac{1}{z^4}\right). \quad (5.10)$$

Unfolding the transformations  $Y \mapsto T \mapsto S \mapsto R$  at large  $z$ , we obtain the identity

$$m^Y = \sqrt{2n} e^{nl\sigma_3/2} (2n)^{n\sigma_3/2} (m^R + m^\infty) (2n)^{-n\sigma_3/2} e^{-nl\sigma_3/2}. \quad (5.11)$$

Since we can reformulate the RH problem for  $R$  in terms of an integral equation

$$R_-(z) = I + \frac{1}{2\pi i} \int_{\Gamma_R} \frac{R_-(z') (G_R(z') - I)}{z' - z} dz', \quad (5.12)$$

we have

$$m^R = -\frac{1}{2\pi i} \int_{\Gamma_R} R_-(z') (G_R(z') - I) dz'. \quad (5.13)$$

Next we iterate the integral equation to find an asymptotic expansion for  $R_-$  as  $n \rightarrow \infty$ . Given that integration over the contours other than  $\partial U^1$  gives only a  $O(n^{-1})$  contribution (because the jump matrix  $G_R = I + O(n^{-1})$  on  $\Gamma_R \setminus \partial U^1$ ), the large  $n$  expansion for  $m^R$  is

$$m^R = -\frac{1}{2\pi i} \oint_{\partial U^1} (G_R(z') - I) dz' - \frac{1}{2\pi i} \oint_{\partial U^1} \rho_1(z') (G_R(z') - I) dz' + O(n^{-1}). \quad (5.14)$$

We can now substitute the asymptotics (5.3) to get, after a straightforward calculation,

$$m^R = n^{-1/3} \operatorname{res}_{z=1} G_1(z) - n^{-2/3} \operatorname{res}_{z=1} G_2(z) + n^{-2/3} \operatorname{res}_{z=1} G_1(z) \cdot \operatorname{res}_{z=1} \frac{G_1(z)}{z-1} + O(n^{-1}), \quad (5.15)$$

as  $n \rightarrow \infty$ . Now, from (5.4) and the expansions (3.17), (3.18) we find that, as  $n \rightarrow \infty$ ,

$$\operatorname{res}_{z=1} G_1(z) = \frac{im_{21}^\Phi(t)}{2} \begin{pmatrix} 1 & -ie^{i\pi\beta} \\ -ie^{-i\pi\beta} & -1 \end{pmatrix} + O(n^{-2/3}) \quad (5.16)$$

and

$$\operatorname{res}_{z=1} \frac{G_1(z)}{z-1} = \frac{im_{21}^\Phi(t)}{10} \begin{pmatrix} 1 & -ie^{i\pi\beta} \\ -ie^{-i\pi\beta} & -1 \end{pmatrix} + O(n^{-2/3}), \quad (5.17)$$

as well as, from (5.5),

$$\operatorname{res}_{z=1} G_2(z) = \frac{im_{11}^\Phi(t)}{2} \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix} + O(n^{-2/3}). \quad (5.18)$$

Note that  $\operatorname{res}_{z=1} G_1$  is nilpotent, thus the third term in (5.15) is negligible. Furthermore, from the relations (3.27) and (3.31), we find

$$m_{11}^\Phi = \frac{1}{2} (m_{21}^\Phi)^2 - \frac{i}{2} (m_{21}^\Phi)'. \quad (5.19)$$

Substituting all previous results into (5.15) we obtain the final formula

$$m^R = \frac{im_{21}^\Phi(t)}{2} \begin{pmatrix} 1 & -ie^{i\pi\beta} \\ -ie^{-i\pi\beta} & -1 \end{pmatrix} n^{-1/3} - \frac{im_{11}^\Phi(t)}{2} \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix} n^{-2/3} + O(n^{-1}). \quad (5.20)$$

The second matrix in (5.11),  $m^\infty$ , can be easily found from (3.12):

$$m^\infty = \frac{i}{2} \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix}. \quad (5.21)$$



All the operations performed to obtain the asymptotics of  $m^R$  from the asymptotics of  $\Phi$  preserve the uniformity in  $\tau \in [\tau_0, \infty)$ , or equivalently  $t \in [t_0, \infty)$ , for any  $\tau_0, t_0 \in \mathbb{R}$ .

By substituting  $m^R$  and  $m^\infty$  into (5.11) we find the large  $n$  expansion for  $R_n$ ,

$$\begin{aligned} R_n &= m_{12}^Y m_{21}^Y = 2n \left( m_{12}^R + m_{12}^\infty \right) \left( m_{21}^R + m_{21}^\infty \right) = \\ &= 2n \left( \frac{m_{21}^\Phi}{2} n^{-1/3} - \frac{im_{11}^\Phi}{2} n^{-2/3} + \frac{i}{2} + O(n^{-1}) \right) \left( \frac{m_{21}^\Phi}{2} n^{-1/3} + \frac{im_{11}^\Phi}{2} n^{-2/3} - \frac{i}{2} + O(n^{-1}) \right), \end{aligned} \quad (5.22)$$

which, by (5.19), simplifies to

$$R_n(\lambda_0; \beta) = \frac{n}{2} - \frac{y(t; \beta)}{2} n^{1/3} + O(1) \text{ as } n \rightarrow \infty, \text{ for all } t \in \mathbb{R}, \text{ uniformly for } t \geq t_0, \quad (5.23)$$

since  $(-im_{21}^\Phi)' = y$ . This result holds for all  $t \in \mathbb{R}$  and  $\beta$  such that  $|\operatorname{Re} \beta| < 1/2$ . This asymptotic series formally matches the classical Hermite recurrence coefficient asymptotics when  $\lambda > 1$  ( $t \rightarrow +\infty$ ) and the non-critical asymptotics from [21] when  $\lambda < 1$  ( $t \rightarrow -\infty$ ).

### 5.3 Plancherel-Rotach type formula and asymptotics for $Q_n$

We can express the orthogonal polynomial  $p_n$  in terms of the RH solution  $Y$ ,

$$p_n(\lambda \sqrt{2n}) = Y_{11}(\lambda \sqrt{2n}) = \lim_{z \rightarrow \lambda} (2n)^{n/2} S_{11}(z) e^{ng_+(z)}, \quad (5.24)$$

where the limit for  $S$  is taken for  $z$  approaching  $\lambda = 1 + \frac{i}{2}n^{-2/3}$  from the upper half plane, outside the lens-shaped region in Fig. 1. If  $z$  lies in this region and  $z \in U^1$  (the small disk around 1 in which the local parametrix  $P^{(1)}$  was constructed), then we can unwind the transformations  $S \mapsto P^{(1)} \mapsto \Phi \mapsto \Psi_0$  to obtain

$$S(z) = R(z) P^{(1)}(z) = R(z) E(z) \begin{pmatrix} 1 & -\frac{i\tau^2}{4} \\ 0 & 1 \end{pmatrix} \Psi_0(\zeta(z) - \tau) e^{-i\pi\beta\sigma_3/2} e^{\frac{2}{3}\zeta(z)^{3/2}\sigma_3}. \quad (5.25)$$

In order to compute the limit  $z \rightarrow \lambda$ , we need to use the small  $\xi$  expansion of  $\Psi_0$  in sector  $I$  given in (3.23). After a straightforward calculation, using also (3.44) and (3.18), we get

$$\lim_{z \rightarrow \lambda} S_{11}(z) e^{ng_+(z)} = in^{1/6} c(t) e^{-\frac{2}{3}is + ng_+(\lambda)} \left( 1 + O(n^{-1/3}) \right), \quad n \rightarrow \infty. \quad (5.26)$$

Therefore,

$$p_n(\lambda \sqrt{2n}) = (2n)^{n/2} e^{-\frac{2}{3}is + ng_+(\lambda)} ic(t) n^{1/6} \left( 1 + O(n^{-1/3}) \right), \quad n \rightarrow \infty. \quad (5.27)$$

From the Lax pair identity (3.34) and  $y(t; \beta) = u(t; \kappa)^2$ ,

$$(ic(t; \beta))^2 = \frac{2\pi u(\tau; \kappa)^2}{\kappa^2} \quad (5.28)$$

and

$$ic(t; \beta) = \pm \frac{\sqrt{2\pi} u(\tau; \kappa)}{\kappa}. \quad (5.29)$$

The right hand side does not depend on the sign of  $\kappa$  (indeed, changing  $\kappa$  to  $-\kappa$  changes  $u$  to  $-u$ ), and we can verify which sign is correct using the asymptotics for  $c$  as  $t \rightarrow \infty$ . Since

$$c(\tau) = \lim_{z \rightarrow 0} (\Psi_0(z))_{21},$$

working backwards along the transformations  $\Psi_0 \mapsto A \mapsto B \mapsto C \mapsto D$  for both  $\tau \rightarrow \pm\infty$ , we can easily recover the asymptotics for  $c(\tau)$ . It turns out that

$$ic(\tau; \beta) = \frac{e^{-\frac{2}{3}\tau^{3/2}}}{\sqrt{2}\tau^{1/4}} \left(1 + O(\tau^{-2})\right) \text{ as } \tau \rightarrow +\infty, \quad (5.30)$$

which implies that the correct sign in (5.29) is +.

*Remark 12.* In the special case  $\beta = 0$ , the model RH problem for  $\Psi_0$  reduces to the Airy model RH problem. In this case, we have

$$ic(\tau; \beta = 0) = \sqrt{2\pi} \text{Ai}(\tau), \quad (5.31)$$

which is indeed consistent with (5.30).

We thus have

$$p_n(\lambda \sqrt{2n}) = \frac{\sqrt{2\pi}}{\kappa} (2n)^{n/2} e^{-\frac{2}{3}is + ng_+(\lambda)} u(t; \kappa) n^{1/6} \left(1 + O(n^{-1/3})\right), \quad n \rightarrow \infty. \quad (5.32)$$

Now we need the expansion of  $g(z)$  near  $z = 1$ :

$$g(z) = \frac{1}{2} - \log 2 + 2(z-1) - \frac{2}{3}2^{3/2}(z-1)^{3/2} + O(z-1)^2 \text{ as } z \rightarrow 1. \quad (5.33)$$

Substituting  $\lambda = 1 + \frac{t}{2}n^{-2/3}$ , we have

$$2ng_+(\lambda) = n - 2n \log 2 + 2tn^{1/3} + \frac{4}{3}is + O(n^{-1/3}) \text{ as } n \rightarrow \infty. \quad (5.34)$$

This gives us the asymptotics as  $n \rightarrow \infty$  of the polynomials  $p_n$  near the critical point,

$$p_n(\lambda \sqrt{2n}) = \frac{\sqrt{2\pi}}{\kappa} \left(\frac{ne}{2}\right)^{n/2} n^{1/6} e^{tn^{1/3}} u(t; \kappa) \left(1 + O(n^{-1/3})\right). \quad (5.35)$$

By multiplying the recurrence relation (1.4) by  $p_n(x)w(x)$  and integrating, we find

$$Q_n = -h_n^{-1} p_n(\lambda \sqrt{2n})^2 e^{-2n\lambda^2} \sinh(i\pi\beta). \quad (5.36)$$

Note that

$$\begin{aligned} h_n &= -\lim_{z \rightarrow \infty} 2\pi i Y_{21}(z \sqrt{2n}) (z \sqrt{2n})^{n+1} = -\lim_{z \rightarrow \infty} 2\pi i \sqrt{2n} (2n)^n e^{nl} z S_{12}(z) = \\ &= -2\pi i \sqrt{2n} (2n)^n e^{nl} (m_{12}^\infty + m_{12}^R), \end{aligned} \quad (5.37)$$

thus the following large  $n$  asymptotics hold for the normalizing coefficients  $h_n$

$$h_n = \frac{\pi \sqrt{2nn^n}}{2^n e^n} e^{i\pi\beta} \left(1 - im_{21}^\Phi(t) n^{-1/3} - m_{11}^\Phi(t) n^{-2/3} + O(n^{-1})\right). \quad (5.38)$$

This proves (1.26). Equivalently, this result can be deduced from the identity

$$h_n = \frac{H_{n+1}}{H_n}, \quad (5.39)$$

expressing  $h_n$  as a ratio of two Hankel determinants, together with the asymptotics (1.13). Substituting (5.38) and (1.27) in (5.36), we obtain (1.25).

Lastly, we note that we can easily obtain the asymptotics of the coefficients in (5.38) as  $t \rightarrow -\infty$ . Formally this can be done by computing an antiderivative of the asymptotics of  $y(t)$ . The following asymptotics were obtained rigorously by solving the RH problem in the limit  $t \rightarrow -\infty$ . For  $|\operatorname{Re} \beta| < 1/2$ ,

$$\begin{aligned} -im_{21}^\Phi(\tau; \beta) = & -2i\beta \sqrt{-\tau} - \frac{1}{4i(-\tau)} \left( \frac{\Gamma(1-\beta)}{\Gamma(\beta)} e^{i\theta(\tau; \beta)} - \frac{\Gamma(1+\beta)}{\Gamma(-\beta)} e^{-i\theta(\tau; \beta)} \right) - \\ & - \frac{3\beta^2}{2(-\tau)} + O\left(\frac{1}{(-\tau)^{5/2-3|\operatorname{Re} \beta|}}\right), \text{ as } \tau \rightarrow -\infty. \end{aligned} \quad (5.40)$$

When  $\beta = i\kappa$ ,  $\kappa \in \mathbb{R}$ , this becomes

$$\begin{aligned} -im_{21}^\Phi(\tau; i\kappa) = & 2\kappa \sqrt{-\tau} + \frac{\kappa}{2(-\tau)} \cos\left(\frac{4}{3}(-\tau)^{3/2} + 3\kappa \log(-\tau) + 6\kappa \log 2 - 2 \arg \Gamma(i\kappa)\right) + \\ & + \frac{3\kappa^2}{2(-\tau)} + O\left(\frac{1}{(-\tau)^{5/2}}\right), \text{ as } \tau \rightarrow -\infty. \end{aligned} \quad (5.41)$$

When  $\beta = 1/2 + i\gamma$ ,  $\gamma \in \mathbb{R}$ , we have

$$-im_{21}^\Phi\left(\tau; \frac{1}{2} + i\gamma\right) = \sqrt{-\tau} \left(2\gamma - \operatorname{tg}\left(\frac{\tilde{\theta}}{2}\right)\right) + O\left(\frac{1}{\tau}\right), \text{ as } \tau \rightarrow -\infty. \quad (5.42)$$

These formulas complement Theorem 5.

## 6 Hankel determinants: alternative proof of Theorem 2

### 6.1 Differential identity

Here, we derive a differential identity for the logarithm of the Hankel determinant  $H_n(\lambda_0, \beta)$ . It is expressed in terms of  $Y$  defined in (3.3).

**Proposition 13.** *We have*

$$\frac{\partial}{\partial \lambda_0} \log H_n(\lambda_0, \beta) = \frac{1}{\pi} \sin \pi \beta \left(Y^{-1} Y'\right)_{21}(\lambda_0) e^{-\lambda_0^2}. \quad (6.1)$$

Here  $'$  is the derivative of  $Y(z)$  with respect to  $z$ .

*Proof.* We write  $P_k = \kappa_k p_k$ ,  $\kappa_k = \frac{1}{\sqrt{h_k}} > 0$  for the normalized orthogonal polynomials with respect to the weight  $w$ . We start from the general identity (equation (17) in [27])

$$\frac{\partial}{\partial \lambda_0} \log H_n(\lambda_0, n) = -n \frac{\kappa_{n-1}}{\kappa_{n-1}} + \frac{\kappa_{n-1}}{\kappa_n} (J_1 - J_2), \quad (6.2)$$

where

$$J_1 = \int_{\mathbb{R}} \dot{P}_n(x) P'_{n-1}(x) w(x) dx, \quad (6.3)$$

$$J_2 = \int_{\mathbb{R}} P'_n(x) \dot{P}_{n-1}(x) w(x) dx. \quad (6.4)$$

Here and below dots denote  $\lambda_0$ -derivatives and primes denote  $x$ -derivatives.

To compute  $J_1$ , we proceed as follows: by (6.3) and (1.2), we have

$$J_1 = \frac{\partial}{\partial \lambda_0} \left( \int_{\mathbb{R}} P_n(x) P'_{n-1}(x) w(x) dx \right) - \int_{\mathbb{R}} P_n(x) \dot{P}'_{n-1}(x) w(x) dx + 2i \sin(\pi\beta) P_n(\lambda_0) P'_{n-1}(\lambda_0) e^{-\lambda_0^2}. \quad (6.5)$$

The first two terms vanish by orthogonality, and we obtain

$$J_1 = 2i \sin(\pi\beta) (P_n P'_{n-1})(\lambda_0) e^{-\lambda_0^2}. \quad (6.6)$$

Similarly, by (6.4) and (1.2), we have

$$J_2 = \frac{\partial}{\partial \lambda_0} \left( \int_{\mathbb{R}} P'_n(x) P_{n-1}(x) w(x) dx \right) - \int_{\mathbb{R}} \dot{P}'_n(x) P_{n-1}(x) w(x) dx + 2i \sin(\pi\beta) P'_n(\lambda_0) P_{n-1}(\lambda_0) e^{-\lambda_0^2}. \quad (6.7)$$

Using the orthogonality relations, we can compute the first two terms and we get

$$J_2 = -n \frac{\kappa_n}{\kappa_{n-1}^2} \kappa_{n-1} + 2i \sin(\pi\beta) (P'_n P_{n-1})(\lambda_0) e^{-\lambda_0^2}. \quad (6.8)$$

Substituting (6.6) and (6.8) into (6.2), we get

$$\frac{\partial}{\partial \lambda_0} \log H_n(\lambda_0, \beta) = \frac{2i\kappa_{n-1}}{\kappa_n} (P_n P'_{n-1} - P'_n P_{n-1})(\lambda_0) \sin(\pi\beta) e^{-\lambda_0^2} \quad (6.9)$$

$$= \frac{2i}{h_{n-1}} (p_n p'_{n-1} - p'_n p_{n-1})(\lambda_0) \sin(\pi\beta) e^{-\lambda_0^2}, \quad (6.10)$$

and using (3.3), we obtain (6.1).  $\square$

## 6.2 Asymptotics for the logarithmic derivative of $H_n(\lambda_0, \beta)$

Let  $\lambda_0$  be of the form (1.7). The results in Section 3 are valid in the limit where  $n \rightarrow \infty$ , uniformly for  $t \geq t_0$  for any fixed  $t_0 \in \mathbb{R}$ .

Inverting the transformations  $Y \mapsto T$  and  $T \mapsto S$  from Section 3.1, it follows from (6.1) that

$$\frac{\partial}{\partial \lambda_0} \ln H_n(\lambda_0, \beta) = \frac{\sin(\pi\beta)}{\pi \sqrt{2n}} \lim_{z \rightarrow \lambda} \left( S^{-1}(z) S'(z) \right)_{21} \quad (6.11)$$

and the limit is taken in the region outside the lens, see Fig. 1. By (3.6), (3.8) and (3.9) we have  $2g_+(\lambda) - 2\lambda^2 - l = 2h(\lambda)$ , hence

$$\frac{\partial}{\partial \lambda_0} \ln H_n(\lambda_0, \beta) = \frac{\sin \pi\beta}{\pi \sqrt{2n}} \lim_{z \rightarrow \lambda} \left( S^{-1}(z) S'(z) \right)_{21} e^{2nh(\lambda)}.$$

Near  $\lambda$ , we have  $S = RP^{(1)}$ , and this implies

$$\begin{aligned} \frac{\partial}{\partial \lambda_0} \ln H_n(\lambda_0, \beta) &= \frac{1}{\sqrt{2n}} \frac{1}{\pi} \sin \pi \beta \left( \left( P^{(1)} \right)^{-1} \left( P^{(1)} \right)' \right)_{21}(\lambda) e^{2nh(\lambda)} \\ &\quad + \frac{1}{\sqrt{2n}} \frac{1}{\pi} \sin \pi \beta \left( \left( P^{(1)} \right)^{-1} R^{-1} R' P^{(1)} \right)_{21}(\lambda) e^{2nh(\lambda)}. \end{aligned} \quad (6.12)$$

Since  $R$  is close to  $I$ , the second term on the right hand side is small. Using the asymptotics (3.44) for  $R$ , we obtain

$$\frac{\partial}{\partial \lambda_0} \ln H_n(\lambda_0, \beta) = \frac{1}{\sqrt{2n}} \frac{1}{\pi} \sin \pi \beta \left( \left( P^{(1)} \right)^{-1} \left( P^{(1)} \right)' \right)_{21}(\lambda) e^{2nh(\lambda)} + O(n^{-1/2}) e^{2nh(\lambda)}, \quad (6.13)$$

as  $n \rightarrow \infty$ , uniformly for  $t \geq t_0$ . To compute the remaining matrix entry, we can use (3.15), which yields

$$\begin{aligned} \frac{\partial}{\partial \lambda_0} \ln H_n(\lambda_0, \beta) &= \zeta'(\lambda) \frac{1}{\sqrt{2n}} e^{-i\pi\beta} \frac{1}{\pi} \sin \pi \beta \left( \Psi_0^{-1} \Psi_0' \right)_{21}(0) \\ &\quad + \frac{1}{\sqrt{2n}} \frac{1}{\pi} \sin \pi \beta \left( \Phi^{-1}(\tau) E^{-1}(\lambda) E'(\lambda) \Phi(\tau) \right)_{21}(\lambda) + O(n^{-1/2}) e^{2nh(\lambda)}, \end{aligned} \quad (6.14)$$

as  $n \rightarrow \infty$ . By (3.19), the second term in the right hand side is of order  $O(n^{-1/6} e^{-\tau})$  uniformly for  $t \geq t_0$ . The first term will be larger than the last two: by (3.17), we get

$$\frac{\partial}{\partial \lambda_0} \log H_n(\lambda_0, \beta) = \sqrt{2} n^{1/6} e^{-i\pi\beta} \frac{1}{\pi} \sin \pi \beta \left( \Psi_0^{-1} \Psi_0' \right)_{21}(0) + O(n^{-1/6} e^{-\tau}), \quad (6.15)$$

as  $n \rightarrow \infty$ , uniformly for  $t \geq t_0$ . Write

$$r(\tau) := \left( \Psi_0^{-1} \Psi_0' \right)_{21}(0; \tau). \quad (6.16)$$

Then, as  $n \rightarrow \infty$ ,

$$\frac{\partial}{\partial \lambda_0} \log H_n(\lambda_0, \beta) = \sqrt{2} n^{1/6} e^{-i\pi\beta} \frac{1}{\pi} \sin(\pi\beta) r(\tau) + \Delta(n, t), \quad (6.17)$$

where  $\Delta(n, t) = O(n^{-1/6} e^{-\tau})$  uniformly for  $t \geq t_0$  as  $n \rightarrow \infty$ .

### 6.3 Expression for $r$ in terms of $u$

**Proposition 14.** *Let  $r$  be defined by (6.16),  $\Psi_0$  as introduced in Section 3.3, and let  $u$  be the Painlevé II solution characterized by (1.9). The following identity holds,*

$$\frac{\partial}{\partial \tau} r(\tau; \beta) = \frac{-2\pi i}{1 - e^{-2i\pi\beta}} u(\tau; \kappa)^2, \quad (6.18)$$

where  $\kappa$  and  $\beta$  are related by (1.16).

*Proof.* Define

$$\hat{\Psi}_0(\xi) = \begin{pmatrix} 1 & -m_{21} \\ 0 & 1 \end{pmatrix} \Psi_0(\xi). \quad (6.19)$$

This transformation has the advantage that it simplifies the  $\tau$ -equation in the Lax pair (3.28), (3.29). We have

$$\left( \frac{\partial}{\partial \tau} \hat{\Psi}_0 \right) \hat{\Psi}_0^{-1} = -i\xi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} 0 & w \\ -1 & 0 \end{pmatrix}, \quad (6.20)$$

where  $w$  is some unknown function of  $\tau$ . In what follows, primes will be used for differentiation w.r.t.  $\tau$ . Now, we start from (3.23). In sector I, we can write

$$\Psi_0(\xi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (I + E_1 \xi + O(\xi^2)) \left( I + \frac{\kappa^2}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ln \xi \right), \quad \xi \rightarrow 0, \quad (6.21)$$

for some matrix  $E_1$  which is  $\xi$ -independent. We easily see from (6.16) and (6.19) that

$$r(\tau) = \left( \Psi_0^{-1} \Psi'_0 \right)_{21} (0; \tau) = \left( \hat{\Psi}_0^{-1} \hat{\Psi}'_0 \right)_{21} (0; \tau) = E_{1,21}(\tau). \quad (6.22)$$

Substituting (6.21) in (6.20), we obtain

$$E'_{1,21}(\tau) = ic^2(\tau). \quad (6.23)$$

By (5.28), we have

$$E'_{1,21}(\tau) = -\frac{2\pi i}{\kappa^2} u(\tau; \kappa)^2. \quad (6.24)$$

Together with (6.22) and (6.23), this implies (6.18).  $\square$

## 6.4 Proof of Theorem 2

As  $n \rightarrow \infty$ , we have  $\tau \sim t$ , see (3.18). Integrating (6.17) from  $\lambda_0 = \sqrt{2n}(1+t_0 n^{-2/3}/2)$  to  $\lambda_1 = \sqrt{2n}(1+t_1 n^{-2/3}/2)$  with  $t_0 < t_1$ , we obtain

$$\begin{aligned} \ln H_n(\sqrt{2n}(1 + \frac{t_0}{2} n^{-2/3}), \beta) - \ln H_n(\sqrt{2n}(1 + \frac{t_1}{2} n^{-2/3}), \beta) = \\ -e^{-i\pi\beta} \frac{1}{\pi} \sin \pi\beta \int_{t_0}^{t_1} r(\tau) d\tau + \frac{1}{\sqrt{2}} n^{-1/6} \int_{t_0}^{t_1} \Delta(n, t) dt. \end{aligned} \quad (6.25)$$

We note that here, as well as in (6.17),

$$\tau = \zeta \left( 1 + \frac{t}{2} n^{-2/3} \right).$$

Writing the left hand side of this expression in an explicit form, one can easily check that there exists a positive constant  $c_0$  such that  $\tau \geq c_0 t$  for all  $t > 1$  and all  $n > 1$ . Hence we can let  $t_1 \rightarrow +\infty$  in (6.25) and, taking into account that  $e^{-i\pi\beta n} H_n(\sqrt{2n}(1 + t_1 n^{-2/3}/2), \beta)$  tends to the Gaussian Hankel determinant  $H_n(\lambda_0, 0)$  without the jump, arrive at the estimate

$$\log H_n(\sqrt{2n}(1 + \frac{t_0}{2} n^{-2/3}), \beta) - \log H_n(\lambda_0, 0) - i\pi\beta n = -e^{-i\pi\beta} \frac{1}{\pi} \sin \pi\beta \int_{t_0}^{\infty} r(\tau) d\tau + O(n^{-1/3}), \quad (6.26)$$

or

$$H_n(\sqrt{2n}(1 + \frac{t_0}{2} n^{-2/3}), \beta) = -e^{i\pi\beta n} H_n(\lambda_0, 0) \exp \left( -\frac{e^{-i\pi\beta} \sin \pi\beta}{\pi} \int_{t_0}^{\infty} r(\tau) d\tau \right) (1 + o(1)), \quad (6.27)$$

as  $n \rightarrow \infty$ . We note that we have replaced  $dt$  with  $d\tau$  in the integral  $\int_{t_1}^{\infty} r(\tau) d\tau$ . This is justified in the limit  $n \rightarrow \infty$  since  $\tau = t(1 + O(tn^{-2/3}))$  and because  $r(\tau)$ , being proportional to an integral of  $u(\tau; \kappa)^2$ , decays exponentially at positive infinity. Finally, substituting (6.18) into this expression and integrating by parts (keeping in mind the above mentioned decay of  $r(\tau)$ ), we obtain (1.13).



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